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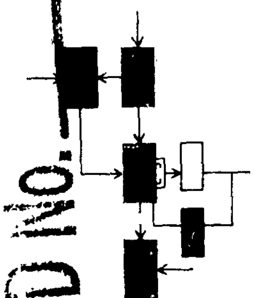

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A RADAR THEORY APPLICABLE *to* DENSE SCATTERER DISTRIBUTIONS

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A RADAR THEORY APPLICABLE TO DENSE SCATTERER DISTRIBUTIONS

by

Arthur Krinitz

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ABSTRACT

Radar is considered as a means of producing a map in two coordinates (x and y) of a dense distribution of scatterers. It is assumed that a complex unit-scatterer return $h(t;x,y)$ and a complex scatterer-density function $\psi(x,y)$ can be defined such that: (a) $h(t;x,y)$ depends only on the radar; (b) $\psi(x,y)$ depends only on the scatterer distribution; (c) the complex video representation of the echo signal is

$s(t) = \int_{-\infty}^{\infty} \psi(x,y)\, h(t;x,y)\, dxdy$; and (d) an x-y display of

$\psi(x,y)$ approximates, in some practically useful way, a map of the distribution of scattering objects in the x-y plane. The above conditions are satisfied if x and y are taken to be time delay and Doppler frequency. Determination of $\psi(x,y)$ is assumed to be the objective of the radar.

A series expansion of $h(t;x,y)$ is obtained in terms of a set of functions, $\Phi_n(t)$, which are orthonormal over the receiver operating time interval T, and a set of functions $\theta_n(x,y)$, which are orthonormal over a finite region R of the x-y plane. If the only *a priori* information is that $\psi(x,y)$ is zero outside the region R, then, even in the absence of noise, only the component of $\psi(x,y)$ which is representable as a linear combination of the $\theta_n(x,y)$ can be deduced from the radar return.

A smoothed form $\psi_s(x,y)$ of $\psi(x,y)$ is defined as the convolution of $\psi(x,y)$ with a spike-like smoothing function. In the presence of additive, stationary, white, Gaussian noise, a maximum likelihood estimate of $\psi_s(x,y)$ can be obtained provided the $\theta_n(x,y)$ are sufficiently complete to permit expanding the desired smoothing function in terms of the $\theta_n(x,y)$. The appropriate receiver is not, in general, a matched filter. The cross section of the smoothing-function spike must be chosen to compromise between loss of detail in $\psi_s(x,y)$ and reduction of the signal-to-noise ratio.

In time delay and Doppler-frequency coordinates, let $\psi_T(x,y)$ be the component of $\psi(x,y)$ that contributes to the radar return during the receiver operating time interval T. If it is known *a priori* that $\psi_T(x,2\pi k/T)$ is zero for x outside

an interval D and for $2\pi k/T$ outside an interval B, and if the transmitted waveform is periodic with $P = D$, then, in the absence of noise, $\psi_T(x, 2\pi k/T)$ can be deduced exactly from the received waveform if, and only if, $BD < 2\pi$ and all harmonics are present in the transmitted waveform.

If the smoothing along the time-delay axis is specified by a spike-like function $a(x)$ for all Doppler frequencies, and if $a(x)$ is a realizable autocorrelation function, then, for a given transmitted average power, the output signal-to-noise ratio is a maximum when the time autocorrelation function of the transmitted waveform is a periodic string of pulses of the form of $a(x)$. The receiver weighting function appropriate for such a waveform resembles closely the matched filter.

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CHAPTER I

INTRODUCTION

A. DELINEATION OF THE AREA OF RESEARCH

The purpose of this research is the development of a radar theory applicable to dense scatterer distributions. The scatterer distribution must be representable, in the sense described in Chap. II, Sect. A, by a scatterer-density function of two coordinates x and y. These could be time delay and Doppler frequency, or any other coordinate pair satisfying the conditions specified in Chap. II. Whether or not other such coordinate pairs exist is not considered in this report.

B. CRITIQUE OF EXISTING MULTIPLE-TARGET RADAR THEORY

A fairly complete theory of detection and coordinate estimation is available for the case of a single point scatterer.[1,2,3,4,5,6*] However, in most radar applications of interest, there is more than one scatterer present at any one time; and often the problem of resolving or separating any one scatterer echo from other scatterer echoes with arbitrary coordinates becomes the overriding consideration in radar design. Resolvability (that is, the degree to which scatterer echoes can be separated), in two

*Superscripts refer to numbered items in the Bibliography.

coordinates, such as time delay and Doppler frequency, is an important problem in multiple-scatterer theory. In experimenting with different transmitted waveforms, it becomes apparent that good resolvability between some targets may be achieved at the expense of resolvability among other targets. Often, for a given transmitted waveform, the resolvability may be improved at the expense of detectability. In spite of these observations, it has, so far, been impossible to obtain a set of necessary and sufficient realizability conditions or limiting relationships which prevent the simultaneous achievement of arbitrary resolvability and detectability for a general class of target distributions. Furthermore, a systematic procedure for the synthesis of the receiver and the transmitted waveform is lacking.

The most useful guide in choosing the transmitted waveform has been Woodward's uncertainty function.[1,2] Siebert[7] has studied the properties of this function to assist the synthesis of the transmitted waveform by obtaining a test of the realizability of any particular form of the uncertainty function. Several necessary conditions were established, but the lack of a useful sufficient condition is still a major obstacle in the synthesis of the transmitted waveform.

C. THE LIMITING FORM OF THE MULTIPLE-TARGET PROBLEM

It appears appropriate, for certain radar applications, to consider dense-scatterer distributions containing scatterers at all points throughout a region of the plane of the coordinates x and y in which the distribution is to be mapped. Under these conditions the detection of any one scatterer and its resolvability from all the other scatterers are not usually of direct interest to the radar user. The user is primarily interested in characteristic patterns formed by clusters or assemblies of scatterers and their over-all distribution in the x-y plane. It is therefore assumed that the purpose of a radar system in this application is to obtain an x-y display of the distribution of scatterers. Consequently, the appropriate measure of performance is the fidelity with which such a display reproduces the actual scatterer distribution.

In applying existing multiple-target radar theory to this dense-scatterer problem, the following difficulties are encountered:

1. The theory is incomplete; it lacks limiting relationships, realizability conditions, and a systematic synthesis procedure.
2. The measures of performance, that is detectability and resolvability, are not directly related to the interests of the radar user.

As a result, a new approach to the dense-scatterer problem has been undertaken.

D. THE SCOPE OF THE PROPOSED THEORY

An appropriate scatterer-density model representing the physical scatterer distribution is developed. The determination of this model can then be regarded as the objective of the radar system in dense-scatterer applications. Limiting relations showing the extent to which the scatterer-density model may be deduced from the radar return are obtained for the noise-free and the noisy cases. A systematic synthesis procedure for the appropriate receiver is developed. Also, the synthesis of transmitted waveforms is carried out for the case in which x is time delay and y is Doppler frequency.

The proposed theory is pertinent to applications involving dense-scatterer distributions. The single-scatterer case has been thoroughly covered by existing radar theory. Between the single-scatterer and the dense-scatterer distributions, there lies a large intermediate class of distributions which may be approximated by a finite number of discrete point scatterers, whose coordinates are not known *a priori*. With respect to these distributions, the dense-scatterer theory, just as the single-scatterer theory, may prove useful as a limiting form of the actual problem.

CHAPTER II

GENERAL THEORY

A. THE RADAR MODEL

The bandwidths of all pertinent radar waveforms (the transmitted waveform, the return from a point scatterer, and the return from an arbitrary distribution of scatterers) are assumed to be less than twice the carrier frequency so that each waveform may be represented as a complex video signal with a one-one relation obtaining between waveforms and their representations. Such a representation does not contain the carrier; only the amplitude and phase modulation are preserved. Suitable linear combinations of the real and imaginary parts of the complex video yield in-phase and quadrature components of actual video signals encountered in coherent radar.

The complex video waveform representing the return from a point scatterer is designated by

$$\text{Return from a point scatterer} = \Psi h(t;x,y) \qquad (2\text{-}1)$$

where $h(t;x,y)$ depends on the radar characteristics (transmitted waveform, antenna pattern, and antenna scan), t is time, x and y are the coordinates of the scatterer, and Ψ is a complex amplitude representing the magnitude and phase of the return.

The return from a number of point scatterers so located that the return from each is independent of the presence of the others is given by a linear superposition of returns of the form of Eq. 2-1; namely,

$$\left.\begin{array}{c}\text{Return from a set of}\\ \text{non-interferring point scatterers}\end{array}\right\} = \sum_k \Psi_k h(t;x_k,y_k) \qquad (2\text{-}2)$$

where Ψ_k, x_k, and y_k are the complex amplitude and the coordinates of $\underline{k}^{th}$ scatterer. For dense-scatterer distributions containing scatterers at all points of the x-y plane, the analog of Eq. 2-2 is:

$$s(t) = \int\int_{-\infty}^{\infty} \Psi(x,y)\; h(t;x,y)\; dx\; dy \qquad (2\text{-}3)$$

where s(t) is the radar return and $\Psi(x,y)$ is the density per unit area of the x-y plane, of the sources of backscatter at a point (x,y). Discrete point scatterers may be accounted for by two-dimensional impulses in $\Psi(x,y)$ at the appropriate points in the x-y plane. Consequently, Eqs. 2-1 and 2-2 may be considered special cases of Eq. 2-3. Thus for noninteracting scatterers, a display of $|\Psi(x,y)|$ is a map of the distribution of physical objects in x-y coordinates--nonzero and zero values of $|\Psi(x,y)|$ indicate, respectively, the presence or absence of objects at the corresponding regions of the x-y plane.

Configurations of objects of practical interest often consist of interacting scatterers which, among other things, may give rise to destructive interference and fading of the target, the appearance of false targets due to multiple reflection, and loss of targets due to shadowing by objects at closer range.

Despite this, it may still be possible that in an appropriate set of coordinates, Eq. 2-3 is valid for a function $h(t;x,y)$ determined by the radar and a function $\Psi(x,y)$ that depends on the scattering objects and that, under the proper conditions (such as those discussed in Sect. A of Chap. III), can be made independent of the radar. It is shown in Sect. A of Chap. III that time delay and Doppler frequency are an appropriate set of coordinates which meet the above requirements.

In consequence, we consider henceforth some coordinate pair x,y for which we are able to define a scatterer-density function having the following important properties:

1. $\Psi(x,y)$ is independent of the radar
2. the received waveform for a radar characterized by any function $h(t;x,y)$ is given by Eq. 2-3
3. $\Psi(x,y)$ characterizes a class of actual scatterer configurations having members which are indistinguishable by a radar characterized by any $h(t;x,y)$ and using any receiver processing
4. $|\Psi(x,y)|$ may sometimes be interpreted, at least approximately, as a distribution of objects throughout the x-y plane

In view of the above properties and because, given $\Psi(x,y)$, $|\Psi(x,y)|$ is available for whatever interpretation as a map of the distribution of objects in the x-y plane may be possible, the desired radar output is assumed in this report to be an x-y display that reproduces $\Psi(x,y)$ within some prescribed fidelity criterion.

We have, in effect, subdivided the dense-scatterer problem into three parts:

1. the determination of a suitable pair of coordinates and a corresponding function $\Psi(x,y)$

2. the interpretation of $|\Psi(x,y)|$ and its relationship to the distribution of objects

3. the establishment of conditions under which the desired radar output is achievable and the determination of the appropriate transmitted waveform, antenna pattern and scan, and receiver processing functions for obtaining the desired output.

Henceforth, aside from Sect. A in Chap. III, our only concern will be part 3 of the dense-scatterer problem.

B. PERFORMANCE LIMITATIONS IN THE ABSENCE OF NOISE

In the following we shall determine the extent to which the scatterer density can be deduced from the knowledge of the noise-free received and transmitted waveforms if the only available a priori information is that $\Psi(x,y)$ vanishes outside some finite region R of the x-y plane. In other words, we shall inquire into the possibility of solving for $\Psi(x,y)$ from Eq. 2-3 when $s(t)$ and the transmitted waveform, and consequently $h(t;x,y)$, are known and the region of integration is restricted to a finite region R.

First, two basic sets of orthonormal functions, one in t and one in x and y, will be obtained. The radar return $s(t)$, the return from a point scatterer $h(t;x,y)$, and the scatterer density $\Psi(x,y)$ will then be represented in terms of the two basic sets. With the help of these representations, the solution to the problem posed above will become obvious. Furthermore, the two basic sets of orthonormal functions will prove very useful in obtaining ultimate performance limitations in the presenoe of noise and in developing a synthesis procedure.

We begin by defining a function $K(t,t')$ as follows:

$$K(t,t') = \iint_R h(t;x,y)\, h^*(t';x,y)\, dx\, dy \qquad (2\text{-}4)$$

where the asterisk denotes the complex conjugate. It is assumed that $|h(t;x,y)|$ is bounded and that the region R is finite. We note that

$$K(t,t') = K^*(t',t)$$

and that, as a consequence of the above assumptions,

$$\int_T \int_T |K(t,t')|^2\, dt\, dt'$$

is finite for any finite T. Thus, $K(t,t')$ is an L_2 (integrable square), Hermitian kernel.[8,9] The corresponding characteristic functions $\varphi_n(t)$ and characteristic values λ_n satisfy the following equations:*

$$\int_T K(t,t')\, \varphi_n(t')\, dt' = \lambda_n\, \varphi_n(t) \qquad n=1,2,3,\ldots;\ t \text{ inside } T \qquad (2\text{-}5)$$

$$\int_T \varphi_m(t)\, \varphi_n^*(t)\, dt = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases} \qquad (2\text{-}6)$$

$$\lambda_n \neq 0 \qquad n=1,2,3\ldots$$

*Most treatments of integral equations consider only real symmetric kernels except that the statement is made that the results can be readily extended to Hermitian kernels as is done here.

The interval of integration T is the observation interval which is a fixed period of time for which the received waveform s(t) (see Eq. 2-3) is available at the radar receiver.

Knowing the characteristic functions $\varphi_m(t)$, we generate a set of functions $\theta_m(x,y)$ as follows:

$$\int_T h^*(t';x,y)\ \varphi_m(t')\ dt' = \mu_m\ \theta_m(x,y) \tag{2-7}$$

where the μ_m are normalizing constants such that

$$\iint_R |\theta_m(x,y)|^2\ dx\ dy = 1 \tag{2-8}$$

Since the normalization can always be accomplished with a positive real constant, we shall take the μ_n to be positive and real.

As a result of Eq. 2-7, we have

$$\int_T h(t;x,y)\ \varphi_n^*(t)\ dt = \mu_n\ \theta_n^*(x,y) \tag{2-9}$$

Multiplying Eq. 2-7 by Eq. 2-9, integrating with respect to x and y, and interchanging the order of integration on the left-hand side, we obtain

$$\int_T\int_T \varphi_m(t')\left[\iint_R h(t;x,y)h^*(t';x,y)dxdy\right]\varphi_n^*(t)dtdt' = \mu_m\mu_n \iint_R \theta_m(x,y)\ \theta_n^*(x,y)dxdy \tag{2-10}$$

Substituting Eq. 2-4 for the bracketed term on the left-hand side and making use of Eqs. 2-5, 2-6, and 2-8, we find that

$$\iint_R \theta_m(x,y)\ \theta_n^*(x,y)\ dx\ dy = \begin{cases} 1 & \text{if}\quad m = n \\ 0 & \text{if}\quad m \neq n \end{cases} \tag{2-11}$$

and that

$$\mu_n^{\;2} = \lambda_n \tag{2-12}$$

Equation 2-12 indicates that $K(t,t')$ is a non-negative kernel (recall that the μ_n's are positive and real), which fact can also be ascertained by noting that

$$\iint_R \left| \int_T h(t;x,y)\; u(t)\; dt \right|^2 dx\; dy \geq 0$$

which with the help of Eq. 2-4 becomes

$$\int_T \int_T u(t)\; K(t,t')\; u^*(t')\; dt\; dt' \geq 0 \tag{2-12a}$$

for any $u(t)$.

It is useful to note that the functions $\varphi_n(t)$ can be generated from the functions $\theta_n(x,y)$ by first multiplying both sides of Eq. 2-7 by $h(t;x,y)$, integrating with respect to x and y, and interchanging the order of integration, so that

$$\mu_n \iint_R h(t;x,y)\theta_n(x,y)dxdy = \int_T \left[\iint_R h(t;x,y)h^*(t';x,y)dxdy\right]\varphi_n(t')dt'$$

Next, by substituting $K(t,t')$ as given by Eq. 2-4 for the bracketed term on the right-hand side and by using Eqs. 2-5 and 2-12, we obtain

$$\iint_R h(t;x,y)\;\theta_n(x,y)\; dx\; dy = \mu_n\; \varphi_n(t) \tag{2-13}$$

In order to obtain a representation for $h(t;x,y)$ in terms of the $\varphi_n(t)$ and $\theta_m(x,y)$, consider the following expression

$$\epsilon_n = \iiint_{T\,R} \left| h(t;x,y) - \sum_{k=1}^{n} \mu_k \varphi_k(t)\ \theta_k^*(x,y) \right|^2 dxdydt \qquad (2\text{-}14)$$

With the help of Eqs. 2-4, 2-13, and 2-12, we find that

$$\epsilon_n = \int_T K(t,t)\ dt - \sum_{k=1}^{n} \lambda_k \qquad (2\text{-}15)$$

It will be assumed that the radar return from a point scatterer, that is, $h(t;x,y)$, is a continuous function of time uniformly in x and y over R -- an assumption which, from a physical standpoint, does not restrict the generality of the analysis. Consequently, $K(t,t')$ as defined by Eq. 2-4 must also be continuous. Furthermore, since $K(t,t')$ is non-negative (see Eq. 2-12a), we can use a corollary of Mercer's theorem (see reference 9, p. 127), which states

$$\sum_{k=1}^{\infty} \lambda_k = \int_T K(t,t)\ dt \qquad (2\text{-}16)$$

Therefore, as n approaches infinity, Eq. 2-15 becomes

$$\lim_{n \to \infty} \epsilon_n = 0$$

which when applied to Eq. 2-14 yields

$$\lim_{n \to \infty} \iiint_{T\,R} \left| h(t;x,y) - \sum_{k=1}^{n} \mu_k\ \varphi_k(t)\ \theta_k^*(x,y) \right| dx\,dy\,dt = 0$$

Thus, we obtain the representation

$$h(t;x,y) = \underset{n \to \infty}{\text{l.i.m.}} \sum_{k=1}^{n} \mu_k\ \varphi_k(t)\ \theta_k^*(x,y) \qquad (2\text{-}17)$$

which is valid regardless of the completeness or incompleteness of the sets $\varphi_k(t)$ and $\theta_k(x,y)$.

Next, we seek the appropriate series representations for the radar return s(t). Multiplying both sides of Eq. 2-3 by $\varphi_k^*(t)$ integrating over T, interchanging the order of integration on the right-hand side, and using Eq. 2-9, we find that

$$s_k = \mu_k \iint_R \psi(x,y)\ \theta_k^*(x,y)\ dx\ dy \tag{2-18}$$

where

$$s_k = \int_T s(t)\ \varphi_k^*(t)\ dt \tag{2-19}$$

Multiplying both sides of Eq. 2-18 by $\varphi_k(t)$ and summing from k=1 to k=n, we obtain

$$\sum_{k=1}^{n} s_k\ \varphi_k(t) = \sum_{k=1}^{n} \mu_k\ \varphi_k(t) \iint_R \psi(x,y)\ \theta_k^*(x,y)\,dx\,dy \tag{2-20}$$

Subtracting Eq. 2-20 from Eq. 2-3 and re-arranging terms gives

$$s(t) - \sum_{k=1}^{n} s_k\varphi_k(t) = \iint_R \left[h(t;x,y) - \sum_{k=1}^{n} \mu_k\varphi_k(t)\theta_k^*(x,y)\right]\psi(x,y)dxdy$$

By applying the Schwarz inequality, we obtain

$$\left|s(t) - \sum_{k=1}^{n} s_k\varphi_k(t)\right|^2 \leq$$

$$\iint_R \left|h(t;x,y) - \sum_{k=1}^{n} \mu_k\varphi_k(t)\theta_k^*(x,y)\right|^2 dxdy \iint_R \left|\psi(x,y)\right|^2 dxdy \tag{2-21}$$

With the help of Eqs. 2-4, 2-12, and 2-13, the first factor on the right-hand side may be rewritten so that

$$\left|s(t) - \sum_{k=1}^{n} s_k\varphi_k(t)\right|^2 \leq \left[K(t,t) - \sum_{k=1}^{n} \lambda_k\varphi_k(t)\varphi_k^*(t)\right] \iint_R \left|\psi(x,y)\right|^2 dxdy \tag{2-21a}$$

Since K(t,t') is continuous and non-negative (see discussion below Eq. 2-15), we may, in proceeding to the limit as n approaches infinity, apply Mercer's theorem to the bracketed expression in Eq. 2-21a. Thus, assuming that $\psi(x,y)$ is an integrable-square function* over the region R, we obtain

$$\lim_{n \to \infty} \left| s(t) - \sum_{k=1}^{n} s_k \, \varphi_k(t) \right|^2 = 0$$

so that

$$s(t) = \sum_{k=1}^{\infty} s_k \, \varphi_k(t) \qquad \text{(2-21b)}$$

converges almost uniformly.[9] Note that Eq. 2-21b is valid regardless of the completeness or incompleteness of the $\varphi_k(t)$.

We are now in a position to inquire into the possibility of solving for $\psi(x,y)$ from Eq. 2-3. For this purpose we define the coefficients ψ_k as follows:

$$\psi_k = \iint_R \psi(x,y) \, \theta_k^*(x,y) \, dx \, dy \qquad \text{(2-22)}$$

where we have used Eq. 2-18. From the theory of orthonormal expansions[9], it is known that $\psi(x,y)$ may be decomposed into two orthogonal components as follows:

$$\psi(x,y) = \psi_{||}(x,y) + \psi_{\perp}(x,y) \qquad \text{(2-23)}$$

such that

*Such a restriction excludes point targets which produce impulses in $\psi(x,y)$. It does not exclude approximations to point targets that produce large-magnitude narrow-base spikes in $\psi(x,y)$.

$$\iint\limits_R \psi_{||}(x,y)\ \psi_{\perp}^*(x,y)\ dx\ dy = 0 \tag{2-23a}$$

$$\psi_{||}(x,y) = \underset{n \to \infty}{l.i.m.} \sum_{k=1}^{n} \psi_k\ \theta_k(x,y) \tag{2-23b}$$

$$\iint\limits_R \left|\psi_{||}(x,y)\right|^2 dx\ dy = \sum_{k=1}^{\infty} \left|\psi_k\right|^2 \tag{2-23c}$$

$$\iint\limits_R \psi(x,y)\ \theta_k^*(x,y)\ dxdy = \iint\limits_R \psi_{||}(x,y)\ \theta_k^*(x,y)\ dxdy = \psi_k \tag{2-23d}$$

$$\iint\limits_R \psi_{\perp}(x,y)\ \theta_k^*(x,y)\ dxdy = 0 \qquad \text{for each } k \tag{2-23e}$$

and

$$\|\psi\| = \|\psi_{||}\| + \|\psi_{\perp}\| \tag{2-23f}$$

where the double vertical bars are used to denote the norm or integrated square of the function shown between the bars. If the system of functions $\theta_k(x,y)$ is complete over the region R, then $\psi_{\perp}(x,y) = 0$.

By substituting Eq. 2-23 into Eq. 2-18 and making use of Eq. 2-23e, we find that $\psi_{\perp}(x,y)$ makes no contribution to any of the s_k. Thus, from Eq. 2-21b, it follows that

$$\iint\limits_R h(t;x,y)\ \psi_{\perp}(x,y)\ dx\ dy = 0 \tag{2-24}$$

Consequently, the component $\psi_{\perp}(x,y)$, which will be referred to as the ambiguous component of the scatterer density, does not contribute anything to the radar return; and the most that we can hope to determine from $s(t)$ is the unambiguous component

$\Psi_{||}(x,y)$. The latter can indeed be determined by a radar receiver which operates on $s(t)$ with a weighting function $W_n(x,y;t)$ defined by

$$W_n(x,y;t) = \sum_{k=1}^{n} \frac{1}{\mu_k} \varphi_k^*(t)\ \theta_k(x,y) \tag{2-25}$$

so that the result of the operation is

$$\int_T W_n(x,y;t)\ s(t)\ dt = \sum_{k=1}^{n} \Psi_k\ \theta_k(x,y) \tag{2-26}$$

where we have used Eq. 2-3, 2-9, and 2-22. In the limit as n approaches infinity, we obtain the representation for $\Psi_{||}(x,y)$ given by Eq. 2-23b.

Recall that $K(t,t')$ is a non-negative L_2 kernel so that its characteristic values λ_n, and hence the μ_n (see Eq. 2-12), may be arranged in a sequence of nonincreasing values[8,9] with increasing n. Consequently, if there is an infinite number of μ_n's, the weighting function given by Eq. 2-25 does not tend to a limit as $n \to \infty$. However, the result of operating with this weighting function (that is, Eq. 2-26) does have a limit in the mean by Eq. 2-23b.

In the foregoing we have obtained a representation of $h(t;x,y)$ in terms of its characteristic functions $\varphi_n(t)$ and $\theta_n(x,y)$ and its characteristic values μ_n. The scatterer density has been decomposed into two orthogonal components $\Psi_{||}(x,y)$ and $\Psi_{\perp}(x,y)$ such that the first is representable as a linear combination of the characteristic functions of $h(t;x,y)$; whereas, the second is not. Thus, the radar return, which was shown to contain contributions only from the characteristic functions of

$h(t;x,y)$ carries no information about $\Psi_{\perp}(x,y)$. As a result, only $\Psi_{\parallel}(x,y)$ can be deduced from the radar return. For this reason $\Psi_{\parallel}(x,y)$ and $\Psi_{\perp}(x,y)$ are referred to, respectively, as the unambiguous and the ambiguous components of the scatterer density. If $h(t;x,y)$ and the region R are such that the characteristic functions $\theta_n(x,y)$ are complete, then, in the absence of noise, a series representation for the total scatterer density $\Psi(x,y)$ can be deduced from the radar return.

The completeness condition may be restated in a number of equivalent forms.[8,9] However, there is no direct mathematical test for explicitly establishing the completeness or incompleteness of the $\theta_n(x,y)$ for any $h(t;x,y)$. Thus, the problem of deducing $\Psi(x,y)$ from $s(t)$ has been reduced to a well-known mathematical problem.

The indexing of the characteristic functions $\theta_k(x,y)$ is arranged in order of nonincreasing μ_k or λ_k which, in general, can be assumed to correspond to increasing "wiggliness" of the $\theta_k(x,y)$ as a function of x and/or y. Thus, high-order (high-k) characteristic functions represent the sharp features of $\Psi_{\parallel}(x,y)$; whereas, low-order (low-k) characteristic functions represent the d-c - like, or smooth, features of $\Psi_{\parallel}(x,y)$.

The incompleteness of a set of characteristic functions $\theta_k(x,y)$ may be regarded as a deficiency of orthogonal functions required for the representation of $\Psi_{\perp}(x,y)$. The set of missing functions can also be classified into high- and low-order functions. Deficiencies of the first class result in an error in the sharpness of detail on the radar display (which can only show $\Psi_{\parallel}(x,y)$), and deficiences of the second class result in an error in the smooth features on the display.

It is clear from Eq. 2-23 that ignorance of $\psi_{\perp}(x,y)$ will also make it impossible to obtain a display of $|\psi(x,y)|$. Qualitatively, the deficiencies in a display of $|\psi_{||}(x,y)|$ are the same as the deficiencies in $\psi_{||}(x,y)$ discussed above.

C. SMOOTHING AND ESTIMATION OF THE SCATTERER DENSITY IN THE PRESENCE OF NOISE

In the presence of noise, which is assumed to be Gaussian, stationary, white, and additive, it becomes impossible to make an exact determination of $\psi_{||}(x,y)$ or $|\psi_{||}(x,y)|$; much less, of $\psi(x,y)$ or $|\psi(x,y)|$. It will be shown that, under these conditions, only an estimate of a smoothed form of $\psi_{||}(x,y)$ can be obtained. Such behavior is, to a certain extent, analogous to that encountered in the problem of extracting an unknown signal s(t) from a background of white, Gaussian, additive noise. In order to obtain an undistorted replica of s(t), an infinite bandwidth is required which, in turn, results in infinite noise power. Therefore, a certain degree of smoothing and the resulting signal distortion is accepted in order to reduce the noise power. The optimum smoothing operation (that is, one that minimizes some measure of the total error caused by signal distortion and noise) depends on the characteristics of s(t) and can be determined only from <u>a priori</u> knowledge about s(t). Since such information is not available for $\psi(x,y)$, we shall consider the effect of a class of smoothing operations upon both signal, in our case $\psi(x,y)$, and noise.

One example of a linear smoothing operation is given by:

$$\Psi_s(x,y) = \iint_R S(x-x',y-y')\,\Psi(x',y')\,dx'\,dy' \qquad (2\text{-}27)$$

where $\Psi_s(x,y)$ is the result of smoothing and $S(x,y)$ has the form of the right cylinder illustrated in Fig. 1a. The cross-section area of the cylinder is σ, the height is $1/\sigma$, and the volume is one. For such an $S(x,y)$, $\Psi_s(x,y)$ in Eq. 2-27 is the average of $\Psi(x,y)$ over the area σ.

More generally $|S(x,y)|$ may have any spike-like form, such as the function illustrated in Fig. 1b. The result of smoothing, in this more general case, can be thought of as a weighted average of $\Psi(x,y)$, if the volume under $S(x,y)$ is set equal to one.

$$\iint_{-\infty}^{\infty} S(x,y)\,dx\,dy = 1 \qquad (2\text{-}28)$$

It is convenient to define the effective averaging area of a general spike-like $S(x,y)$ as the cross section σ of a cylinder that is equivalent to the spike in the sense that the volumes under the squared magnitude of the two functions are equal. Thus,

$$\sigma = \frac{1}{\iint_{-\infty}^{\infty} |S(x,y)|^2\,dx\,dy} \qquad (2\text{-}29)$$

Of particular importance in the present context is the class of smoothing functions for which (a) $S(x,y)$ falls off rapidly for x and y outside an area σ about $x = 0$, $y = 0$; and (b) σ is much smaller than the area of the region R. For such a smoothing function, the radar display contains significant detail.

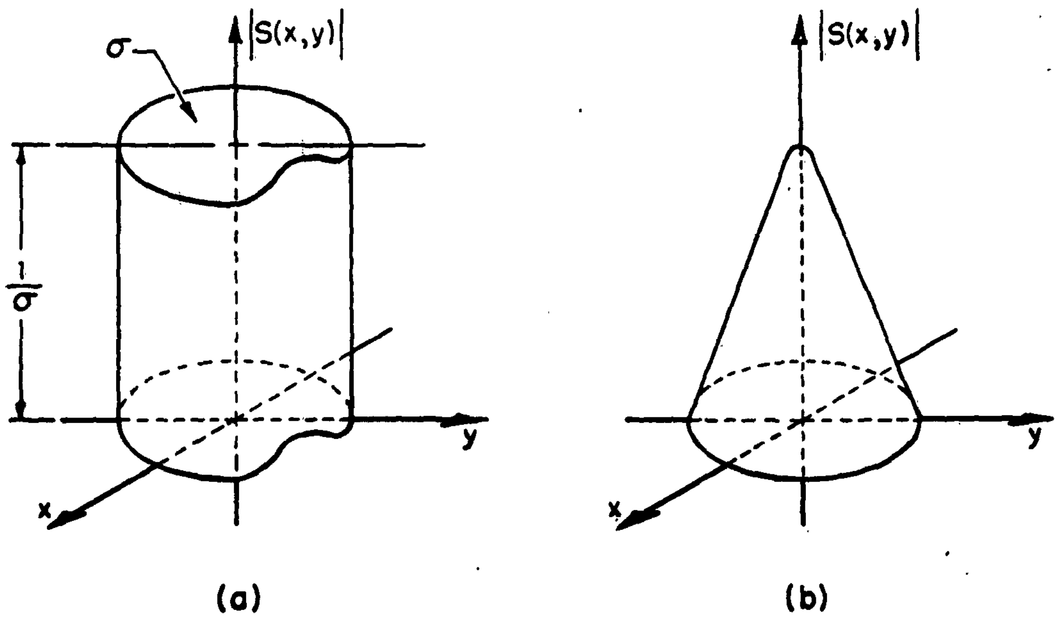

Fig. 1 Examples of Smoothing Functions

If we regard $S(x-x',y-y')$ as a function of x' and y' with x and y as parameters, we can write

$$S(x-x',y-y') = \sum_n p_n(x,y)\ \theta_n^*(x',y') + S_\perp(x,y;x',y') \qquad (2\text{-}30)$$

where

$$p_n(x,y) = \iint_R S(x-x',y-y')\ \theta_n(x',y')\ dx'dy' \qquad (2\text{-}30a)$$

and $S_\perp(x,y;x',y')$ is the component of $S(x-x',y-y')$ orthogonal to all the functions $\theta_n(x',y')$,

$$\iint_R S_\perp(x,y;x',y')\ \theta_n(x',y')\ dx'dy' = 0 \qquad (2\text{-}30b)$$

Substitution of Eq. 2-30 into Eq. 2-27 and use of Eqs. 2-22, 2-23, 2-23b, and 2-30b yield

$$\Psi_s(x,y) = \sum_k \Psi_k p_k(x,y) + \iint_R S_\perp(x,y;x',y')\,\Psi_\perp(x',y')\ dx'dy' \qquad (2\text{-}31)$$

Since only the Ψ_k's can be deduced from the received waveform (even under noise-free conditions), the last term in Eq. 2-31 constitutes an ambiguous component of $\Psi_s(x,y)$. One means of obtaining unambiguous determination of $\Psi_s(x,y)$ in the noise-free case, therefore, is to select a smoothing function and transmitted waveform in such relationship to one another that $S_\perp(x,y;x',y')$ is zero. In these circumstances,

$$\Psi_s(x,y) = \sum_k \Psi_k\ p_k(x,y) \qquad (2\text{-}31a)$$

The radar return from which $\Psi_s(x,y)$ is to be estimated is given by:

$$r(t) = s(t) + n(t) \qquad (2\text{-}32)$$

where s(t) is the signal component, given by Eq. 2-3, and n(t) is the noise component in the complex video representation.[4] Multiplying both sides of Eq. 2-32 by $\varphi_k{}^*(t)$ and integrating with respect to t over the interval T gives

$$r_k = s_k + n_k \qquad k = 1, 2, 3, \ldots \tag{2-33}$$

where

$$r_k = \int_T r(t)\ \varphi_k{}^*(t)\ dt \tag{2-33a}$$

s_k is given by Eq. 2-19, and

$$n_k = \int_T n(t)\ \varphi_k{}^*(t)\ dt \tag{2-33b}$$

With the help of Eq. 2-22, Eq. 2-33 may be rewritten as follows:

$$r_k = \mu_k \Psi_k + n_k \tag{2-34}$$

Equation 2-33b is to be interpreted as a stochastic integral[10]; and n_k, a random variable.

Since the s_k's completely specify the signal component s(t) (see Eq. 2-19), the r_k's may be used as the "observables" or "observable coordinates."[3,4] Thus, the problem of estimating $\Psi_s(x,y)$ from r(t) may be restated as a problem of estimating $\Psi_s(x,y)$ from the r_k's.

It was assumed that the noise n(t) is stationary, white, Gaussian noise of zero mean; let its spectral density be N_0. With the help of Eq. 2-33b, we can evaluate the following averages:

$$\overline{n_k\, n_\ell^*} = \int_T\int_T \overline{n(t)\; n^*(t')}\; \varphi_k(t)\; \varphi_\ell^*(t')\; dt\; dt'$$

$$= \int_T\int_T N_0\; \delta(t-t')\; \varphi_k(t)\; \varphi_\ell^*(t')\; dt\; dt'$$

$$\overline{n_k\, n_\ell^*} = N_0 \int_T \varphi_k(t)\; \varphi_\ell^*(t)\; dt \tag{2-35}$$

where the horizontal bar denotes an ensemble average. Thus, using Eq. 2-6, we obtain

$$\overline{n_k\, n_\ell^*} = \begin{cases} N_0 & \text{if} \quad k = \ell \\ 0 & \text{if} \quad k \neq \ell \end{cases} \tag{2-36}$$

Since linear operations on Gaussian random processes result in Gaussian random processes or variables,[4,10] all the n_k's are jointly Gaussian random variables with zero mean. Thus, for a given set of ψ_k's, the r_k's as given by Eq. 2-34 are jointly Gaussian random variables whose joint probability density will be denoted by

$$\text{Probability density} = P(r_1, r_2, r_3, \ldots / \psi_1, \psi_2, \psi_3, \ldots) \tag{2-37}$$

From Eqs. 2-34 and 2-37 and from the fact that the average value of n_k is zero, we conclude that the maximum-likelihood estimate ψ_{ke} of ψ_k is given by

$$\psi_{ke} = \frac{r_k}{\mu_k} \tag{2-38}$$

If $S(x-x', y-y')$ is nonsingular, that is, if given $\psi_s(x,y)$, Eq. 2-30 is satisfied by a unique function $\psi(x,y)$; and furthermore if $S_\perp(x,y;x',y')$ is zero, then a one-to-one correspondence holds between sets of ψ_k and functions

$\Psi_s(x,y)$, as given by Eq. 2-31a. Under these circumstances the maximum-likelihood estimate of $\Psi_s(x,y)$ is formally given by

$$\Psi_{se}(x,y) = \sum_{k=1}^{\infty} \Psi_{ke}\, p_k(x,y) \qquad (2\text{-}39)$$

where $\Psi_{se}(x,y)$ is the estimate of $\Psi_s(x,y)$. Using Eqs. 2-38, 2-34, and 2-31a in Eq. 2-39 and redefining terms, we obtain

$$\Psi_{se}(x,y) = \Psi_s(x,y) + n(x,y) \qquad (2\text{-}40)$$

where $n(x,y)$ is given by

$$n(x,y) = \sum_{k=1}^{\infty} \frac{n_k}{\mu_k}\, p_k(x,y) \qquad (2\text{-}40a)$$

Thus, according to Eq. 2-40, the final output on the radar display consists of two components, one of which is the smoothed scatterer density and the other is the noise. From Eqs. 2-40, 2-40a, 2-30a, and 2-27, it follows that

$$\Psi_{se}(x,y) = \iint_R S(x-x',y-y')\,\Psi(x',y')\,dx'dy'$$

$$+ \sum_{k=1}^{\infty} \frac{n_k}{\mu_k}\left[\iint_R S(x-x',y-y')\;\theta_k(x',y')\,dx'dy'\right] \qquad (2\text{-}41)$$

If $S(x-x',y-y'')$ is singular--that is, if knowledge of $\Psi_s(x,y)$ does not determine $\Psi(x,y)$ uniquely, but still $S_{\perp}(x,y;x',y')$ is zero--the set of Ψ_k's cannot be deduced from $\Psi_s(x,y)$. Observe, however, that: (1) an estimate $\Psi_{se}(x,y)$ is determined by Eq. 2-41, whether or not $S(x-x',y-y')$ is singular; and (2) as the input noise decreases to zero, the n_k's tend to zero and $\Psi_{se}(x,y)$ approaches $\Psi_s(x,y)$, whether $S(x-x',y-y')$ is singular or not. We therefore consider Eq. 2-41 to yield a reasonable estimate of $\psi_s(x,y)$ in the singular case.

If $S_\perp(x,y;x',y')$ is not zero, the set of $\theta_k(x',y')$ is not complete; and for some $\psi(x',y')$, $\psi_\perp(x',y')$ not only is not zero but also is not orthogonal to $S_\perp(x,y;x',y')$ for some (x,y). By Eq. 2-31, therefore, an ambiguous component of $\psi_s(x,y)$ exists even in the noise-free case. We, therefore, conclude that a meaningful estimate (given by Eq. 2-41) can be obtained if, and only if, the set of $\theta_k(x,y)$'s is sufficiently complete so that $S(x-x',y-y')$, regardless of whether it is singular or not, is representable in the form of Eq. 2-30b.

The mean-square value of the noise at any point on the display is given by

$$\overline{|n(x,y)|^2} = N_0 \sum_{k=1}^{\infty} \frac{|p_k(x,y)|^2}{\mu_k^2} \tag{2-42}$$

where we have made use of Eqs. 2-30a and 2-36 in Eq. 2-40a. If we average the mean-square noise over the area of the display, that is, the region R, we obtain the average mean-square noise

$$N_R = \frac{1}{R} \iint_R \overline{|n(x,y)|^2}\, dxdy = N_0 \sum_{k=1}^{\infty} \frac{\overline{p_k^2}}{\mu_k^2} \tag{2-43}$$

where $\overline{p_k^2}$ is the average value of $|p_k(x,y)|^2$; namely,

$$\overline{p_k^2} = \frac{1}{R} \iint_R |p_k(x,y)|^2\, dxdy \tag{2-43a}$$

In applications in which $|\psi(x,y)|$ is a meaningful representation of the scatterer distribution (see Sect. A of this chapter), the radar user may be primarily interested in the magnitude of

$\Psi_s(x,y)$. Unfortunately the relation between $|\Psi_s(x,y)|$ and the set of Ψ_k's is not one-to-one so that a maximum-likelihood estimate of $|\Psi_s(x,y)|$ cannot be made. We shall, therefore, use as our estimate of the magnitude of $\Psi_s(x,y)$ the magnitude of the estimate of $\Psi_s(x,y)$, namely $|\Psi_{se}(x,y)|$.

D. NOISE AND SMOOTHING

In this section a performance index which is an average-signal-to-average-noise ratio is defined, and expressions for this ratio are obtained in terms of useful system parameters.

The energy received from a unit point scatterer is given by

$$E(x,y) = \int_T \left|h(t;x,y)\right|^2 dt \tag{2-44}$$

which, in general, is a function of x and y. The average energy over the region R is given by

$$E_0 = \frac{1}{R} \iint_R \int_T \left|h(t;x,y)\right|^2 dt\,dx\,dy \tag{2-45}$$

where R is used to designate both the region occupied by scatterers and its area.

It follows from Eqs. 2-17 and 2-45 that

$$E_0R = \sum_{k=1}^{\infty} \mu_k^2 \tag{2-46}$$

Furthermore, if $S_\perp(x,y;x',y') = 0$, it follows from Eq. 2-30 that

$$\iint_R \left|S(x-x',y-y')\right|^2 dx'dy' = \sum_{k=1}^{\infty} \left|p_k(x,y)\right|^2$$

for x and y inside R. Since the cross section of $S(x,y)$ is much smaller than the area R, we can rewrite the above expression as follows:

$$\iint_{-\infty}^{\infty} \left| S(x,y) \right|^2 \, dx \, dy = \sum_{k=1}^{\infty} \overline{p_k^{\,2}}$$

where $\overline{p_k^{\,2}}$ is given by Eq. 2-43a. Making use of Eq. 2-29, we obtain

$$\sum_{k=1}^{\infty} \overline{p_k^{\,2}} = \frac{1}{\sigma} \tag{2-47}$$

It is convenient to define the following sets of normalized quantities

$$\alpha_k = \frac{\overline{p_k^{\,2}}}{\sum_{n=1}^{\infty} \overline{p_n^{\,2}}} \tag{2-48}$$

and

$$\rho_k = \frac{\mu_k^{\,2}}{\sum_{n=1}^{\infty} \mu_n^{\,2}} \tag{2-49}$$

Applying Eqs. 2-46 through 2-49 to Eq. 2-43 and redefining terms gives

$$N_R = \frac{N_0}{E_0} \frac{J}{\sigma R} \tag{2-50}$$

where

$$J = \sum_{k=1}^{\infty} \frac{\alpha_k}{\rho_k} \tag{2-51}$$

Because σ is much smaller than R, $\Psi_s(x,y)$ is negligible outside R except in a region that (for regions R of reasonable shape) is negligible relative to R. Thus, the average signal intensity

$\frac{1}{R} \iint_R \left|\Psi_s(x,y)\right|^2 dx\,dy$ is very closely approximated by

$$S = \frac{1}{R} \iint_{-\infty}^{\infty} \left|\Psi_s(x,y)\right|^2 dx\,dy \qquad (2\text{-}52)$$

and the average-signal-to-average-noise ratio is

$$\eta = \frac{S}{N_R} \qquad (2\text{-}53)$$

Substituting Eq. 2-27 into Eq. 2-52 and interchanging orders of integration gives

$$S = \frac{1}{R} \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \Psi_R(x',y')K_{ss}(x'-x'',y'-y'')\Psi_R{}^*(x'',y'')dx'dy'dx''dy'' \qquad (2\text{-}54)$$

where

$$K_{ss}(x'-x'',y'-y'') = \iint_{-\infty}^{\infty} S(x-x',y-y')\; S^*(x-x'',y-y'')\; dxdy \qquad (2\text{-}54a)$$

and where

$$\Psi_R(x,y) = \begin{cases} \Psi(x,y) & \text{for } (x,y) \text{ inside } R \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}54b)$$

Changing variables and integrating over x' and y' in Eq. 2-54 gives

$$S = \frac{1}{R} \iint_{-\infty}^{\infty} K_{\psi\psi}(u,v)\; K_{ss}(u,v)\; du\,dv \qquad (2\text{-}55)$$

where

$$K_{\psi\psi}(u,v) = \iint_{-\infty}^{\infty} \Psi_R(x',y')\; \Psi_R{}^*(x'-u,y'-v)\; dx'dy' \qquad (2\text{-}55a)$$

which is the autocorrelation function of the scatterer density.

Two classes of scatterer densities for which explicit results can be obtained are:

1. the class of smooth scatterer densities-- all scatterer densities whose autocorrelation function, $K_{\psi\psi}(u,v)$, does not change appreciably over a region of size σ centered at $u = 0$ and $v = 0$;

2. the class of rough scatterer densities-- all scatterer densities whose autocorrelation function, $K_{\psi\psi}(u,v)$, in the vicinity of $u = 0$ and $v = 0$ is substantially confined to a region much smaller than σ. (The behavior of $K_{\psi\psi}(u,v)$ for u and v outside an area σ about the origin has negligible effect on S--see Eq. 2-55.)

If we note that $K_{ss}(u,v)$ is a spike-like form whose cross section is approximately σ (see Eq. 2-54a) and if we use the subscripts s and r to refer to the smooth and rough class, respectively, then Eq. 2-55 becomes

$$S_s = \frac{1}{R} K_{\psi\psi}(0,0) \iint_{-\infty}^{\infty} K_{ss}(u,v)\, du\, dv \tag{2-56}$$

for smooth scatterer densities, and

$$S_r = \frac{1}{R} K_{ss}(0,0) \iint_{-\infty}^{\infty} K_{\psi\psi}(u,v)\, du\, dv \tag{2-57}$$

for rough scatterer densities. With the help of Eqs. 2-28, 2-29, 2-54a, 2-55a, and 2-54b, Eqs. 2-56 and 2-57 may be rewritten as follows:

$$S_s = \frac{1}{R} \iint_R |\psi(x,y)|^2\, dx\, dy \tag{2-58}$$

and

$$S_r = \frac{R}{\sigma} \left| \frac{1}{R} \iint_R \psi(x,y)\, dx\, dy \right|^2 \tag{2-59}$$

Substituting Eqs. 2-50, 2-58, and 2-59 into Eq. 2-53 and redefining terms, we obtain for the smooth and rough scatterer densities, respectively,

$$\eta_s = \frac{E_0}{N_0}\,\frac{R\sigma}{J}\,\overline{|\Psi|^2} \tag{2-60}$$

and

$$\eta_r = \frac{E_0}{N_0}\,\frac{R^2}{J}\,|\overline{\Psi}|^2 \tag{2-61}$$

where $\overline{|\Psi|^2}$ is the average value of the squared scatterer density and $|\overline{\Psi}|^2$ is the square of the average value of the scatterer density; that is,

$$\overline{|\Psi|^2} = \frac{1}{R}\iint_R |\Psi(x,y)|^2\,dx\,dy \tag{2-62}$$

and

$$|\overline{\Psi}|^2 = \left|\frac{1}{R}\iint_R \Psi(x,y)\,dx\,dy\right|^2 \tag{2-63}$$

In order to provide an interpretation for J in Eqs. 2-60 and 2-61, suppose, for a moment, that all the μ_k's are equal for $1 \leq k \leq M$ and are zero otherwise. The sums in Eqs. 2-30, 2-48, 2-49, and 2-51 then are over, at most, M terms, and J is equal to M--the number of normal modes contained in h(t;x,y). Next, suppose that the values of the μ_k's are approximately constant for $1 \leq k \leq M$ and decrease very rapidly for $k > M$. If α_k has any significant value for $k > M$, then J will be large and η_s and η_r will be small (see Eqs. 2-60 and 2-61). As a result, the α_k's must be confined substantially to $1 \leq k \leq M$ and then, J is again approximately equal to M. Thus, in general, J may be interpreted as the effective number of modes that may be utilized without undue decrease in the average-signal-to-average-noise ratio. Despite the infinite number of normal modes in $h(t;x,y)$, the number of modes that are usable to obtain $\Psi_{se}(x,y)$ is finite because the high-order modes contain little energy and are submerged in noise.

If the smoothed scatterer density is to be an exact replica of the actual scatterer density, then (a) $S(x-x',y-y')$ must be a two-dimensional impulse $\delta(x-x',y-y')$; and (b) the scatterer density $\Psi(x,y)$ must contain no ambiguous component $\Psi_{\perp}(x,y)$. In order for condition (b) to obtain for all L_2 functions $\Psi(x,y)$, the set of $\theta_k(x,y)$ must be complete. In this event the impulse may be regarded as the limit approached by the sequence of functions

$$\sum_{k=1}^{n} \theta_k(x,y)\ \theta_k^*(x',y')$$

because for a complete set of $\theta_k(x,y)$

$$\underset{n\to\infty}{\ell.i.m.} \iint_R \Psi(x',y')\left[\sum_{k=1}^{n} \theta_k(x,y)\ \theta_k^*(x',y')\right] dx'dy' = \Psi(x,y)$$

The corresponding sequence of J's may be evaluated from Eq. 2-51, giving

$$J_n = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\rho_k} \qquad n = 1, 2, 3, \ldots$$

Since the ρ_k's are positive, $J_n > \frac{1}{n\,\rho_n}$ or $\rho_n > \frac{1}{n\,J_n}$. For a finite signal energy E_0, the series

$$\sum_{k=1}^{\infty} \mu_k^{\,2}$$

converges by Eq. 2-46. Hence, according to Eq. 2-49,

$$\sum_{n=1}^{\infty} \rho_n = 1$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\, J_n}$$

must also converge. Since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, we conclude that J_n must approach infinity with increasing n. Hence, exact reproduction of an arbitrary scatterer density $\Psi(x,y)$ requires an infinite number of usable modes.

In general, as the form of S(x,y) is adjusted to shrink σ toward zero in an effort to obtain fine resolution, if at the same time ambiguities in the determination of $\Psi_s(x,y)$ are avoided, then J approaches infinity for any spike-like shape S(x,y). By Eqs. 2-60 and 2-61, the average-signal-to-average-noise ratios η_s and η_r each tend to zero in the process. We conclude that the sharper the details of $\Psi(x,y)$ that we attempt to estimate, the lower the average-signal-to-average-noise ratios become. As a result, the resolvable cell size σ must be chosen to compromise between two requirements:

1. σ must be sufficiently small to assure that the essential details of $\Psi(x,y)$ are not smeared out in the smoothing process
2. σ must be sufficiently large to achieve a desired signal-to-noise ratio

E. THE RECEIVER

The input to the radar receiver is r(t), and the output at the end of the observation interval T is a two-dimensional display of $\psi_{se}(x,y)$ (or $|\psi_{se}(x,y)|$). From Eqs. 2-33a, 2-34, 2-30b, 2-40a, and 2-40, it follows that, if

$$W_s(x,y;t) = \underset{n \to \infty}{l.i.m.} \sum_{k=1}^{n} \frac{p_k(x,y)}{\mu_k} \varphi_k^*(t) \qquad (2\text{-}64)$$

for each x and y, then

$$\int_T W_s(x,y;t)\, r(t)\, dt = \psi_{se}(x,y) \qquad (2\text{-}65)$$

Thus, the receiver is described by the linear operation,

$$\int_T W_s(x,y;t)\, r(t)\, dt \quad .$$

if $\psi_{se}(x,y)$ is the desired output. As long as J is not infinite and σ is not zero, N_R (see Eq. 2-50) is finite so that, as a result of Eq. 2-43, $\overline{|n(x,y)|^2}$ is finite almost everywhere in R. Thus, comparing Eq. 2-64 with Eq. 2-42, we see that the right-hand side of Eq. 2-64 converges in the mean as a function of t, x, and y.

For any $\theta_n(x,y)$ that is nearly constant over the averaging region of the smoothing function S(x,y), Eq. 2-30a yields $p_n(x,y) \approx \theta_n(x,y)$. Thus, if the $\theta_n(x,y)$'s are arranged in order of decreasing smoothness, the leading terms in Eq. 2-64 will resemble the corresponding terms for the noiseless-case receiver given by Eq. 2-25. For a given $|\overline{\psi}|^2$ (or $\overline{|\psi|^2}$) and a lower bound on η_r (or η_s), Eqs. 2-60 and 2-61 indicate that the lower the input noise level N_0 or the greater the signal energy E_0, the closer to an impulse can $S(x,y)$ be made. Thus, as the input noise level decreases, more and more of the terms in Eq. 2-64 will resemble terms of Eq. 2-25 for the noiseless-case receiver.

In order to relate the above results to the conventional theory of point-scatterer detection by radar, we consider a point target at (x_0, y_0). The smoothed scatterer density is, by Eq. 2-27,

$$\Psi_s(x,y) = S(x-x_0, y-y_0)$$

and the signal intensity on the radar display at the position of the target is

$$\left|\Psi_s(x_0,y_0)\right|^2 = \left|\sum_k p_k(x_0,y_0)\ \theta_k^*(x_0,y_0)\right|^2$$

where we have made use of Eq. 2-30. With the help of Eq. 2-42, we obtain the signal-to-noise ratio at the position of the target

$$\frac{\left|\Psi_s(x_0,y_0)\right|^2}{\overline{\left|n(x_0,y_0)\right|^2}} = \frac{1}{N_0} \frac{\left|\sum_k \left[\mu_k \theta_k^*(x_0,y_0)\right]\left[\frac{p_k(x_0,y_0)}{\mu_k}\right]\right|^2}{\sum_k \frac{\left|p_k(x_0,y_0)\right|^2}{\mu_k^2}}$$

Applying the Schwartz inequality to the numerator gives

$$\frac{\left|\Psi_s(x_0,y_0)\right|^2}{\overline{\left|n(x_0,y_0)\right|^2}} \le \frac{1}{N_0} \sum \mu_k^2 \left|\theta_k(x_0,y_0)\right|^2 = \frac{E(x,y)}{N_0}$$

where we have also made use of Eqs. 2-44 and 2-17. The maximum signal-to-noise ratio is achieved when

$$\frac{p_k(x_0,y_0)}{\mu_k} = c(x_0,y_0)\ \mu_k\ \theta_k(x_0,y_0) \qquad (2\text{-}66)$$

where $c(x_0,y_0)$ is an arbitrary function of x_0 and y_0 and is independent of k. Thus, according to Eqs. 2-64 and 2-66, the receiver which maximizes

$$\frac{\left|\Psi_s(x_0,y_0)\right|^2}{\overline{\left|n(x_0,y_0)\right|^2}}$$

for all x_0 and y_0 inside the region R is given by

$$W_0(x,y;t) = c(x,y) \underset{n \to \infty}{\ell.i.m.} \sum_{k=1}^{n} \mu_k \, \theta_k(x,y) \, \varphi_k^*(t) \qquad (2\text{-}67)$$

where $W_0(x,y;t)$ is used to designate the $W_s(x,y;t)$ obtained when $p_k(x,y)$ is given by Eq. 2-66. The resulting smoothing function, which may not have a spike-like form $S(x-x',y-y')$ is given by

$$S(x,y;x',y') = c(x,y) \sum_{k=1}^{\infty} \mu_k^2 \, \theta_k(x,y) \, \theta_k^*(x',y') \qquad (2\text{-}68)$$

Comparing Eqs. 2-67 and 2-68 with Eq. 2-17, we obtain

$$W_0(x,y;t) = c(x,y) \, h^*(t;x,y)$$

and

$$S(x,y;x',y') = c(x,y) \int_T h(t;x,y) \, h^*(t;x',y') \, dt$$

Thus, aside from the multiplying function c(x,y), the receiver reduces to the matched filter; the smoothing function reduces to a generalized form of Woodward's uncertainty function; and the output signal-to-noise power ratio reduces to the energy ratio $E(x,y)/N_0$ of conventional detection theory.

CHAPTER III

DELAY-DOPPLER RADAR THEORY

A. THE SCATTERER-DENSITY MODEL

It is assumed in this chapter that x is time delay and y is Doppler angular frequency. It is further assumed that, if f(t) is the complex-video representation of the transmitted waveform, the radar return from a point scatterer at delay x and Doppler angular frequency y may be expressed as

$$h(t;x,y) = f(t-x)\, e^{-jty} \tag{3-1}$$

As a result of the superposition of electromagnetic fields in linear media, the radar return due to any transmitted wave f(t) can always be represented by

$$s(t) = \int_{-\infty}^{\infty} g(x,t)\, f(t-x)\, dx \tag{3-2}$$

where s(t) is the received waveform and g(x,t) is independent of f(t) and depends only on the antenna, the scatterer distribution, and their relative positions and motion.

On physical grounds it may be assumed that g(x,t) is an integrable-square function of time over any finite time interval L so that the integral

$$\Psi(x,y) = \frac{1}{2\pi} \int_{L} g(x,t)\, e^{jty}\, dt \tag{3-3}$$

exists. Hence, by Fourier transform theory, we have

$$g(x,t) = \int_{-\infty}^{\infty} \psi(x,y)\, e^{-jty}\, dy \qquad (3\text{-}4)$$

for t inside the interval L. Substituting Eq. 3-4 into Eq. 3-2 gives

$$s(t) = \iint_{-\infty}^{\infty} \psi(x,y)\, f(t-x)\, e^{-jty}\, dx\, dy \qquad (3\text{-}5)$$

for t inside L. The interval L is assumed to be sufficiently long so that any observation interval T of interest to the radar user falls inside L and Eq. 3-5 is, therefore, valid for t inside T. Equation 3-5 is the same as Eq. 2-3 if $h(t;x,y)$ is defined by Eq. 3-1. Clearly $h(t;x,y)$ depends only on the radar; and by Eq. 3-3, $\psi(x,y)$, like $g(x,t)$, depends only on the scatterers and the radar antenna. To eliminate dependence of $\psi(x,y)$ on the radar, we assume a priori that the scatterers are confined to a region of uniform, unchanging antenna illumination. Thus, the conditions required of the coordinates x and y in Chap. II, Sect. A, are satisfied and determination of $\psi(x,y)$ can be considered the radar objective in a study of performance obtainable with different transmitted waveforms and receiver processing functions. Note that, in view of the choice of $h(t;x,y)$ (see Eq. 3-1), it is to be expected that, subject to the usual restrictions (such as wide antenna beamwidth, narrow transmitter bandwidth, negligible target acceleration), $|\psi(x,y)|$ may be regarded as an approximate map of the distribution of objects throughout the x-y plane.

B. PROPERTIES OF TRANSMITTED WAVEFORMS

It is assumed in the remainder of this chapter that $\Psi(x,y)$ is known a priori to be zero for x outside an interval D, extending from $x = x_1$ to $x = x_2$. It will be shown that, in consequence, certain classes of waveforms need not be considered in evaluating radar-performance possibilities.

Consider, for a moment, a nonperiodic transmitted waveform f(t) in relation to the time scales for f(t) and s(t) in Fig. 2. Since x_1 is the minimum delay and x_2 is the maximum delay, it follows from Fig. 2 that any transmission outside the interval T_x will make no contribution to s(t) inside the observation interval T (which for convenience is centered at $t = 0$). Thus, the transmitted waveform could just as well have been a periodic waveform of period T_x. It is therefore concluded that only periodic transmitted waveforms need be considered.

Next, suppose that a periodic waveform of period P, which is less than the length of the interval D, is used as the transmitted waveform. Since $f(t) = f(t + P)$, it is clear from Eq. 3-5 that any change in $\Psi(x,y)$ coupled with the negative of that change shifted by P along the x-axis, and such that both changes are confined to D, will not affect the radar return. Hence, if $P < D$, an ambiguous component of the scatterer density will arise. Waveforms of period less than D are, therefore, of no interest if unambiguous determination of $\Psi(x,y)$ is desired. In the following section we consider waveforms of period $P = D$. Attempts to treat waveforms of period $P > D$ have been unsuccessful--it has not been possible to determine whether or not limitations present for $P = D$ can be alleviated for $P > D$.

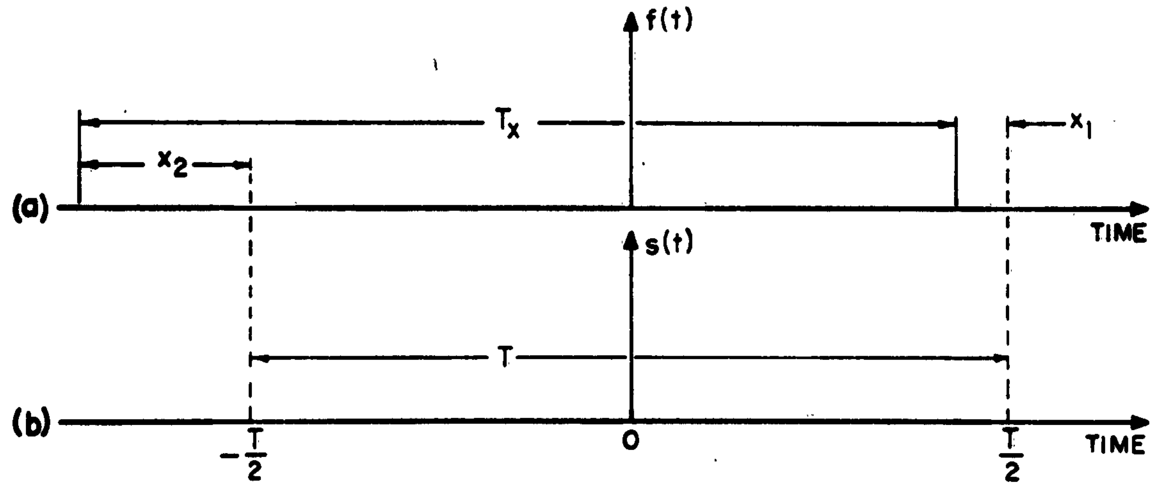

Fig. 2 Axis Showing Time Relationship Between Transmitted and Received Waveforms

C. AMBIGUITIES OF RANGE-DOPPLER RADAR FOR P = D

In this section we consider transmitted waveforms of period P equal to the length of the scatterer-occupied interval D. We shall be interested in establishing conditions under which a sufficiently complete set of $\theta_k(x,y)$'s can be obtained.

The class of waveforms under consideration can be represented by*

$$f(t) = \sum_k f_k \, e^{j2\pi kt/D} \tag{3-6}$$

where the f_k are the Fourier coefficients, and D is the length of the interval D. Furthermore, for t inside the observation interval, we have

$$h(t;x,y) = \sum_k h_k(x,y) \, e^{j2\pi kt/T} \tag{3-7}$$

where

$$h_k(x,y) = \frac{1}{T} \int_T h(t;x,y) \, e^{-j2\pi kt/T} \, dt \tag{3-8}$$

Also note that, without destroying any useful information in the radar return, the observation time T can always be extended so that

$$\frac{T}{D} = I = \text{Positive integer} \tag{3-9}$$

From Eqs. 3-1, 3-6, 3-7, 3-8, and 3-9, it follows that

$$h(t;x,y) = \sum_{n,\ell} f_\ell \, e^{-j2\pi\ell x/D} \, \frac{\sin\left[\pi(I\ell-n) - yT/2\right]}{\pi(I\ell-n) - yT/2} \, e^{j2\pi nt/T} \tag{3-10}$$

Equation 3-5 and 3-10 indicate that any component of the scatterer density which is orthogonal to $\frac{\sin(k\pi - yT/2)}{k\pi - yT/2}$ for all

---

*Henceforth, it is understood that all summation indices run from $-\infty$ to $+\infty$, unless otherwise indicated.

k will make no contribution to the radar returns. Hence, the component that does contribute to the radar return must be expressible as

$$\Psi_T(x,y) = \sum_{k} \Psi_T(x, 2\pi k/T) \frac{\sin(k\pi - yT/2)}{k\pi - yT/2} \qquad (3\text{-}11)$$

It has already been assumed that $\Psi(x,y) = 0$ outside an x interval of length D. It is now assumed also that the samples $\Psi_T(x, 2\pi k/T)$ are known <u>a priori</u> to be zero for $k < k_1$ and $k > k_2$. For $T \gg D$ this assumption is often nearly equivalent to assuming $\Psi(x,y) = 0$ outside a band of Doppler angular frequencies y, extending from $y_1 = 2\pi k_1/T$ to $y_2 = 2\pi k_2/T$. The width of this band is given by

$$B = y_2 - y_1 = 2\pi(k_2 - k_1)/T \qquad (3\text{-}12)$$

Equation 3-11 can, therefore, be rewritten as follows:

$$\Psi_T(x,y) = \sum_{k=k_1}^{k_2} \Psi_T(x, 2\pi k/T) \frac{\sin(k\pi - yT/2)}{k\pi - yT/2} \qquad (3\text{-}13)$$

Henceforth, any scatterer density, whose $\Psi_T(x,y)$ component can be expressed as a finite series in the form of Eq. 3-13 with B determined by Eq. 3-12 and which is zero for x outside D, will be referred to as a BD scatterer density.

Observe that, in the absence of any additional <u>a priori</u> information, $\Psi_T(x,y)$ is the only component of the scatterer density that can possibly be deduced from the radar return. We, therefore, wish to find the maximum value of B, if any, for which, in the absence of noise, $\Psi_T(x,y)$ can be deduced from the radar return. We also wish to determine what restrictions on transmitted waveforms are needed to permit deducing a $\Psi_T(x,y)$ of maximum B.

It is convenient to define a set of coefficients

$$g_k = \begin{cases} f_{k/I} & \text{when } k/I \text{ is an integer} \\ 0 & \text{otherwise} \end{cases} \tag{3-14}$$

and to rewrite Eq. 3-10 in the form

$$h(t;x,y) = h_0(t;x,y) + h_1(t;x,y) \tag{3-15}$$

where

$$h_0(t;x,y) = \sum_n \sum_{m=k_1}^{k_2} g_{n+m} \, e^{-j2\pi(n+m)x/ID} \, \frac{\sin(\pi m - yT/2)}{\pi m - yT/2} \, e^{j2\pi nt/T} \tag{3-16}$$

and $h_1(t;x,y)$ has the same form, but the index m runs over values outside the (k_1,k_2) interval. It follows from Eq. 3-13 that

$$\int_{-\infty}^{\infty} \Psi_T(x,y) \, h_1(t;x,y) \, dy = 0 \tag{3-17}$$

Thus, $h_1(t;x,y)$ is of no concern to us since, for a BD scatterer density, it makes no contribution to the radar return.

Equation 3-16 may be written

$$h_0(t;x,y) = \sum_n \mu_n \, \varphi_n(t) \, \theta_n^*(x,y) \tag{3-18}$$

where

$$\mu_n^2 = 2\pi D \sum_{i=n+k_1}^{n+k_2} |g_i|^2 \tag{3-19}$$

$$\theta_n(x,y) = \frac{\sqrt{T}}{\mu_n} \sum_{m=k_1}^{k_2} g_{m+n}^* \, e^{j2\pi(m+n)x/ID} \, \frac{\sin(\pi m - yT/2)}{\pi m - yT/2} \tag{3-20}$$

and

$$\varphi_n(t) = \frac{1}{\sqrt{T}} \, e^{j2\pi nt/T} \tag{3-21}$$

Note that, if any $\mu_n = 0$ when calculated by Eq. 3-19, then the corresponding term is absent from series, Eq. 3-18, so that there is no need to consider indeterminates in Eq. 3-20. With the help of Eq. 3-14, it can be shown that

$$\iint_R \theta_m(x,y)\ \theta_n^*(x,y)\ dx\ dy = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases} \tag{3-22}$$

where the region R is bounded by the interval D along the x-axis and extends from $-\infty$ to $+\infty$ along the y-axis. Thus, the $\theta_n(x,y)$ form an orthonormal set over R. Furthermore, the $\varphi_n(t)$ form a complete orthonormal set over T.

By Eqs. 3-1 and 3-5, the discussion above Eq. 3-11, Eq. 3-17, and because $\Psi_T(x,y) = 0$ for x outside D, we have

$$s(t) = \int_D \int_{-\infty}^{\infty} \Psi_T(x,y)\ h_0(t;x,y)\ dx\ dy \tag{3-23}$$

for t inside T. Equations 3-23 and 3-18 are analogous to the basic equations of the general theory (Eqs. 2-3 and 2-17). The only change is that the function $\Psi_T(x,y)$, instead of being identically zero outside an interval B (to yield a finite region R), is a sin x/x interpolation of samples spaced at intervals $2\pi/T$ along y; and the sample values are zero outside B. The properties of orthonormal expansions obtain, as in the general theory (see Eqs. 2-22 through 2-23f); therefore, $\Psi_T(x,y)$ is the sum of a component $\Psi_{||}(x,y)$ expressible as a linear combination of the $\theta_n(x,y)$ and an ambiguous component $\Psi_{\perp}(x,y)$ that makes no contribution to s(t) and, hence, cannot be determined by the radar. The necessary and sufficient condition for unambiguous determination of $\Psi_T(x,y)$ is therefore that $\Psi_T(x,y)$ is known *a priori* to be a linear combination of the $\theta_n(x,y)$. We, therefore, wish to determine the maximum value of B for which the $\Psi_T(x,y)$ component of any BD scatterer density can be expanded in terms of $\theta_n(x,y)$.

It is convenient to express each of the samples $\Psi(x, 2\pi k/T)$ in Eq. 3-13 as a Fourier series over the interval D. Hence,

$$\Psi_T(x,y) = \sum_m \sum_{n=k_1}^{k_2} \Psi_{mn}\, u_{mn}(x,y) \tag{3-24}$$

where

$$u_{mn}(x,y) = e^{j2\pi mx/D}\, \frac{\sin(n\pi - yT/2)}{n\pi - yT/2} \tag{3-25}$$

and where the Ψ_{mn} are the appropriate Fourier coefficients. In order to be able to expand $\Psi_T(x,y)$ as a linear combination of the $\theta_k(x,y)$, it is necessary and sufficient that each of the orthogonal functions $u_{mn}(x,y)$ be representable as a linear superposition of the $\theta_k(x,y)$; that is, it is necessary and sufficient that

$$u_{mn}(x,y) = \sum_k u_k^{(mn)}\, \theta_k(x,y) \tag{3-26}$$

where

$$u_k^{(mn)} = \iint_R u_{mn}(x,y)\, \theta_k^*(x,y)\, dx\, dy \tag{3-26a}$$

for all m and for $k_1 \leq n \leq k_2$.

Substituting Eq. 3-26a into Eq. 3-26 and making use of Eqs. 3-25, 3-20, and 3-14, we find that it is necessary and sufficient that

$$u_{mn}(x,y) = 2\pi D\, \frac{g_{Im}}{\mu^2_{Im-n}} \sum_{\ell=k_1}^{k_2} g^*_{Im-n+\ell}\, e^{j2\pi(Im+\ell-n)x/ID}\, \frac{\sin(\pi\ell - yT/2)}{\pi\ell - yT/2} \tag{3-27}$$

Thus, if Eq. 3-26 is to be a correct representation for $u_{mn}(x,y)$

as given by Eq. 3-25, it is necessary and sufficient that

$$\frac{2\pi D}{\mu^2_{Im-n}} g_{Im}\, g^*_{Im+\ell-n} = \begin{cases} 1 & \text{when} \quad \ell = n \\ 0 & \text{for} \quad \ell \neq n,\ k_1 \leq \ell \leq k_2 \end{cases} \tag{3-28}$$

for each m and for $k_1 \leq n \leq k_2$. Implied in Eq. 3-28 is that for no k can $\mu_k = 0$. Observe that, if any $\mu_k = 0$, then the corresponding term would be missing from Eq. 3-26 and the right-hand member of Eq. 3-27 would be zero for the corresponding value of Im-n so that Eq. 3-27 could not be satisfied.

The above necessary and sufficient conditions can be restated as follows:

$$\left|g_{Im}\right| = \frac{\mu_{Im-n}}{\sqrt{2\pi D}} \neq 0 \qquad \text{for each m and } k_1 \leq k \leq k_2 \tag{3-29}$$

and

$$g_{Im}\, g^*_{Im+\ell-n} = 0 \qquad \text{for each m, } k_1 \leq \ell \leq k_2,\ k_1 \leq n \leq k_2,\ \ell \neq n \tag{3-30}$$

If we let $k = \ell - n$, Eq. 3-28 becomes

$$g_{Im}\, g^*_{Im+k} = 0 \qquad \text{for each m, } -(k_2 - k_1) \leq k \leq (k_2 - k_1),\ k \neq 0 \tag{3-31}$$

Equation 3-29 requires that, for all m, $g_{Im} \neq 0$. Therefore, referring to Eq. 3-31, we conclude that, in order to satisfy Eqs. 3-29 and 3-31, it is necessary that

$$k_2 - k_1 < I \tag{3-32}$$

which, with the help of Eqs. 3-9 and 3-12, Eq. 3-32 may be restated as follows:

$$BD < 2\pi \tag{3-33}$$

The maximum value of $k_2 - k_1$, which satisfies Eq. 3-32 is given by

$$k_2 - k_1 = I - 1 \tag{3-34}$$

which value of $k_2 - k$ represents the worst possibility--if an adequate set of $\theta_n(x,y)$ can be achieved for $k_2 - k_1 = I - 1$, then it can also be achieved if fewer samples $\Psi_T(x, 2\pi k/T)$ are permitted by a priori knowledge to differ from zero. Equation 3-34 implies that, for any integer n, a unique integer p can be found such that

$$n + k_1 \leq I_p \leq n + k_2 \tag{3-35}$$

Applying Eq. 3-35 to Eq. 3-19 and making use of Eq. 3-14, we find that

$$\mu_n^{\ 2} = 2\pi D \left|f_p\right|^2 \qquad \text{for} \quad I_p - k_2 \leq n \leq I_p - k_1 \tag{3-36}$$

In other words for every transmitted harmonic f_p, there are $k_2 - k_1 + 1 = I$ equal characteristic values μ_n; and since no $f_p = 0$, then, for every n, $\mu_n \neq 0$. From Eqs. 3-36 and 3-14, it follows that Eq. 3-29 is satisfied. Also, as a result of Eq. 3-34, Eq. 3-30 is satisfied. Thus, we are assured that, if (a) for all p, $f_p \neq 0$; and (b) Eq. 3-33 holds, the set of $\theta_n(x,y)$ is sufficient to expand the $\Psi_T(x,y)$ component of a BD scatterer density, and unambiguous determination of $\Psi_T(x,y)$ is possible. The set of $\theta_n(x,y)$ is obtained by using Eq. 3-35, together with Eq. 3-14 in Eq. 3-20. Hence,

$$\theta_n(x,y) = \sqrt{\frac{T}{2\pi D}}\ \frac{f_p^*}{\left|f_p\right|}\ e^{j2\pi px/D}\ \frac{\sin\left[\pi(I_{p-n}) - yT/2\right]}{\pi(I_{p-n}) - yT/2} \tag{3-37}$$

$$\text{for} \quad I_p - k_2 \leq n \leq I_p - k_1$$

so that, for every transmitted harmonic f_p, there are $k_2 - k_1 + 1 = I$ characteristic functions $\theta_n(x,y)$.

For a transmitted wave of period P = D, we have, therefore, shown that, in order to obtain unambiguous determination of

$\psi_T(x,y)$ for a BD scatterer density, it is necessary and sufficient that

(a) the transmitted wave contains all its harmonics (no $f_k = 0$)

(b) $k_2 - k_1 < I$, or $BD < 2\pi$ -- that is, the scatterer density must be such that the samples $\psi_T(x, 2\pi k/T)$ are zero for $k < k_1$ and $k > k_2$

It is interesting to note that condition b corresponds to the well-known limitation on maximum unambiguous delay and maximum unambiguous Doppler frequency encountered in periodic pulse radar systems.

D. NOISE AND SMOOTHING

It is assumed in this section that Eq. 3-33 is satisfied, that the period of the transmitted waveform is equal to D, and that all harmonics are present. Thus, we are assured that there is no ambiguous component of $\psi_T(x,y)$. In analogy to Sect. D of Chapter II, a performance index which is an average-signal-to-average-noise ratio will be defined. Expressions for this ratio will be obtained in terms of useful system parameters.

Suppose that a smoothing operator of the following form is chosen

$$S(x,y) = a(x)\,b(y) \tag{3-38}$$

where both a(x) and b(y) could be some pulse-like forms centered at $x = 0$ and $y = 0$, respectively. The width of a(x) is much smaller than D, and the width of b(y) is much smaller than B. We can, therefore, write

$$a(x) = \sum_k a_k \, e^{j2\pi kx/D} \tag{3-39}$$

In order to assure that $S(x-x', y-y')$ is expressible in terms of the $\theta_n(x,y)$ (see Chap. II, Sect. C), it is necessary to assume that the Fourier transform of $b(y)$ is zero outside an interval T. Hence,

$$\frac{T}{2\pi} \int_{-\infty}^{\infty} b(y-u) \, \frac{\sin uT/2}{uT/2} \, du = b(y) \tag{3-40}$$

To apply the theory of Chap. II, Sect. C, we note that, in analogy to Eq. 2-30a, we have

$$p_n(x,y) = \int_D \int_{-\infty}^{\infty} S(x-x', y-y') \, \theta_n(x', y') \, dx'dy' \tag{3-41}$$

Thus, using Eqs. 3-38, 3-39, and 3-40 in Eq. 3-41 gives

$$p_n(x,y) = \sqrt{\frac{2\pi D}{T}} \, \frac{f_p^*}{|f_p|} \, a_p \, e^{j2\pi px/D} \, b\left[y - 2\pi(I_p - n)\right] \quad \text{for} \quad I_p - k_2 \leq n \leq I_p - k_1 \tag{3-42}$$

Furthermore, by analogy to Eq. 2-42 and with the help of Eqs. 3-40, 3-42, and 3-36, we obtain

$$\overline{|n(x,y)|^2} = \frac{N_0}{T} \left|\frac{a_p}{f_p}\right|^2 \sum_{m=k_1}^{k_2} \left|b(y - 2\pi m/T)\right|^2 \tag{3-43}$$

The assumption resulting in Eq. 3-40 also yields

$$b(y-u) = \sum_m b(y - 2\pi m/T) \, \frac{\sin(\pi m - uT/2)}{\pi m - uT/2} \tag{3-44}$$

so that

$$\int_{-\infty}^{\infty} |b(y-u)|^2 \, du = \frac{2\pi}{T} \sum_m |b(y - 2\pi m/T)|^2$$

Comparing this expression with Eq. 3-43, we obtain

$$\overline{\left|n(x,y)\right|^2} \leq N_R \tag{3-45}$$

where

$$N_R = \frac{N_0}{2\pi} \sum_p \left|\frac{a_p}{f_p}\right|^2 \int_{-\infty}^{\infty} \left|b(y)\right|^2 dy \tag{3-46}$$

Since b(y) is a narrow spike, much narrower than B, we see that, for y inside B, and not too near the edge of B, we have approximately,

$$\overline{\left|n(x,y)\right|^2} \doteq N_R \qquad \text{for y inside B} \tag{3-47}$$

Furthermore, for y appreciably outside B, $\overline{\left|n(x,y)\right|^2}$ is nearly zero.

The quantity N_R, given by Eq. 3-46, cannot be defined the same way as in the general theory (see Eq. 2-43) because the region R in this chapter is infinite. It is interesting, though, that

$$\frac{1}{BD} \int_D \int_{-\infty}^{\infty} \overline{\left|n(x,y)\right|^2} \, dx \, dy$$

leads exactly to N_R above; and to that extent, BD is the "effective" area of R and N_R is the "effective" average of the mean-square noise.

Because the region R considered here is infinite, the average signal intensity also cannot be defined as in Chap. II (see Eq. 2-52). Nonetheless, the average-signal-to-average-noise ratio calculated from Eqs. 2-42, 2-52, and 2-53,

$$\eta = \frac{\iint_{-\infty}^{\infty} \left|\Psi_s(x,y)\right|^2 dx \, dy}{\iint_R \overline{\left|n(x,y)\right|^2} \, dx \, dy} \tag{3-48}$$

exists inasmuch as the factor 1/R is cancelled in the ratio. By a procedure similar to the one used in obtaining Eq. 2-57, we find that, for a rough scatterer density

$$\iint_{-\infty}^{\infty} \left|\Psi_s(x,y)\right|^2 dxdy = K_{ss}(0,0) \iint_{-\infty}^{\infty} K_{\Psi\Psi}(u,v)\, dudv$$

which, with the help of Eqs. 2-54a and 2-55a, becomes

$$\iint_{-\infty}^{\infty} \left|\Psi_s(x,y)\right|^2 dxdy = \iint_{-\infty}^{\infty} \left|S(x,y)\right|^2 dxdy \left|\iint_R \Psi(x,y)\, dxdy\right|^2 \tag{3-49}$$

A similar relation can be obtained for the smooth case; but we shall confine our attention to the rough case, which appears to be of greater physical significance. It follows from Eqs. 3-43, 3-9, 3-38, and 3-49 that Eq. 3-48 can be written as

$$\eta = \frac{\int_{-\infty}^{\infty} \left|a(x)\right|^2 dx}{N_0 \sum_p \left|a_p/f_p\right|^2} \left|\iint_R \Psi(x,y)\, dxdy\right|^2 \tag{3-50}$$

It is interesting to define the number of useful modes, a quantity analogous to J given by Eq. 2-51 in the general theory. The normalized quantities α_k and p_k can be defined as in Eqs. 2-48 and 2-49, except that $\overline{p_k^2}$ in Eq. 2-48 cannot be defined for an infinite region. However, only the ratio of

$$\overline{p_k^2} \quad \text{to} \quad \sum_n \overline{p_n^2}$$

is needed; hence, we define

$$\alpha_k = \frac{\iint\limits_R \left|p_k(x,y)\right|^2 \, dxdy}{\sum\limits_n \iint\limits_R \left|p_n(x,y)\right|^2 \, dxdy}$$

Making use of Eqs. 2-49, 2-51, 3-36, 3-39, 3-42 and the above definition, we obtain

$$J = I \frac{P_0}{\sum\limits_k \left|a_k\right|^2} \sum_p \left|\frac{a_p}{f_p}\right|^2 \tag{3-51}$$

where P_0 is the average transmitted power given by

$$P_0 = \sum_n \left|f_n\right|^2 \tag{3-52}$$

Note that, as a result of Eqs. 3-1, 3-6, 3-9, and 3-52, we can write

$$\frac{1}{BD} \int\limits_B \int\limits_D \int\limits_T \left|h(t;x,y)\right|^2 \, dt \, dx \, dy = P_0 \, T$$

regardless of the length of the interval B. Comparing the above result with Eq. 2-45, we have

$$E_0 = P_0 \, T \tag{3-53}$$

With the help of Eqs. 3-51, 3-53, and 3-39, we can rewrite Eq. 3-50 as follows:

$$\eta = \frac{E_0}{N_0 \, J} \left|\iint\limits_R \psi(x,y) \, dx \, dy\right|^2 \tag{3-54}$$

which, if we take into account Eq. 2-63, agrees with Eq. 2-61.

E. SYNTHESIS OF THE TRANSMITTED WAVEFORM

In this section we obtain transmitted-waveform requirements which, for a given transmitted average power and the smoothing function specified by Eq. 3-38, yield the maximum output signal-to-noise ratio.

As long as the f_p's are different from zero, the smoothed scatterer density $\psi_s(x,y)$ is independent of the f_p's so that we can choose a set of f_p's to minimize $\overline{|n(x,y)|^2}$, as given by Eq. 3-43 without affecting the signal intensity. Thus, the set of f_p's which minimizes $\overline{|n(x,y)|^2}$ also maximizes both the average-signal-to-average-noise ratio as well as the local signal-to-noise ratio $|\psi_s(x,y)|^2 / \overline{|n(x,y)|^2}$.

It is clear from Eq. 3-43 that the larger the $|f_p|$'s the smaller $\overline{|n(x,y)|^2}$. However, since the average transmitted power P_0 must remain finite, we need to impose Eq. 3-52 as a constraint. Applying the method of Lagrange's multipliers, we minimize Eq. 3-43 under the constraint imposed by Eq. 3-52. The result is

$$|f_n|^2 = \frac{P_0}{\sum_k |a_k|} |a_n| \tag{3-55}$$

Thus, once the smoothing operator has been selected, the optimum distribution of transmitted power throughout the spectrum is determined by Eq. 3-55. The phase angles of the f_n's are arbitrary so that it might be possible, in principle, to minimize the peak power requirements by adjusting the phases.

If a(x) is a realizable autocorrelation function, and if the optimum transmitted waveform is used, then Eq. 3-51 yields

$$J = T \frac{a^2(0)}{\int_{-\infty}^{\infty} |a(x)|^2 \, dx} \tag{3-56}$$

where we have also made use of Eqs. 3-39 and 3-55. The factor $\int_{-\infty}^{\infty} |a(x)|^2 \, dx / a^2(0)$ is a measure of the width of a(x) or a measure of the reciprocal bandwidth of the spectrum of a(x). However, by Eq. 3-55, a(x) is the autocorrelation function of the transmitted wave so that $a^2(0) / \int_{-\infty}^{\infty} |a(x)|^2 \, dx$ is a (somewhat unusual) measure of the bandwidth of the transmitted wave; and, in this special case, J reduces to a radar time-bandwidth product.

The average-signal-to-average-noise ratio for the optimum transmitted waveform is given by Eq. 3-54 with J evaluated as the time-bandwidth product of Eq. 3-56.

F. THE PROCESSING OF THE RADAR RETURN

The receiver weighting function corresponding to the choice of smoothing function given by Eq. 3-38 is obtained in the same way as in the general theory (Chap. II, Sect. F). Equation 2-61 results, except that the summation index now runs from $-\infty$ to $+\infty$. Substituting Eqs. 3-21, 3-36, and 3-41 into Eq. 2-64, we obtain

$$W_s(x,y;t) = \frac{1}{T} \sum_p \frac{a_p}{f_p} e^{j2\pi px/D} \sum_{n=I_p-k_2}^{I_p-k_1} b\left[y - 2\pi(I_p - n)/T\right] e^{-j2\pi nt/T} \tag{3-57}$$

Substituting m for $I_p - n$, making use of Eq. 3-9, and rearranging terms gives

$$W_s(x,y;t) = \left[\sum_p \frac{a_p}{f_p} e^{-j2\pi p(t-x)/D}\right]\left[\frac{1}{T}\sum_{m=k_2}^{m=k_1} b(y-2\pi m/T)\ e^{j2\pi mt/T}\right] \quad (3\text{-}58)$$

It is convenient to define the Fourier transform of b(y)

$$B(t) = \int_{-\infty}^{\infty} b(u)\ e^{-jut}\ du \quad (3\text{-}59)$$

from which it follows that

$$B(t)\ e^{jty} = \int_{-\infty}^{\infty} b(y-v)\ e^{jvt}\ dv \quad (3\text{-}60)$$

Making use of Eq. 3-44 in Eq. 3-60, we obtain

$$B(t)\ e^{-jty} = \frac{2\pi}{T}\sum_m b(y-2\pi m/T)\ e^{j2\pi mt/T} \quad (3\text{-}61)$$

for t inside the interval T. If we recall that the width of the spike-like form b(y) is much narrower than B, we see that, for y inside B, and not too near the edge of B, Eq. 3-61 may be approximately rewritten as follows

$$B(t)\ e^{-jty} = \frac{2\pi}{T}\sum_{m=k_1}^{k_2} b(y-2\pi m/T)\ e^{j2\pi mt/T} \quad (3\text{-}62)$$

Using Eq. 3-62 in Eq. 3-58 gives

$$W_s(x,y;t) = \frac{1}{2\pi}\sum_p \frac{a_p}{f_p} e^{-j2\pi p(t-x)/D}\ B(t)\ e^{-jyt} \quad (3\text{-}63)$$

Suppose that the spike-like form a(x) is a realizable autocorrelation function so that all the a_n are positive. Then, if the optimum transmitted waveform, specified by Eq. 3-55, is used in Eq. 3-63

$$W_s(x,y;t) = \frac{a(0)}{2\pi P_0} B(t)\ f^*(t-x)\ e^{jty} \quad (3\text{-}64)$$

where we have also made use of Eq. 3-39 and Eq. 3-6. If $B(t)$ is constant throughout T, so that $b(y)$ specifies only the unavoidable smoothing associated with a finite observation interval, the weighting function in Eq. 3-60 is the matched filter of conventional radar theory.

G. APPLICATION TO THE MAPPING OF THE LUNAR SURFACE

An interesting application of radar to dense-scatterer distributions is encountered in the problem of radar mapping of the lunar surface in range and range-rate (or delay and Doppler-frequency) coordinates. Such a mapping has been carried out by the Millstone Radar.[11]

Of importance to this mapping is the libration of the moon--an apparent angular vibration of the moon as viewed by an observer on the earth. The major component of libration arises in the following manner: The same hemisphere of the moon is continually pointing toward the center of the earth. Thus, an observer on the surface of the earth sees slightly different portions of the moon as the earth rotates to place the moon in different positions above the observer's horizon. As a result of this effect, the moon appears to rotate on its own axis through a small angle when viewed from a point on the earth's surface. Different points on the surface of the moon will, therefore, have different range rates. Thus, a time-delay and a Doppler-frequency coordinate, together with a scatterer density $\psi(x,y)$, may be assigned to each point on the surface of the moon.

In the Millstone experiment an estimate of $|\psi(x,y)|^2$, where x is time delay and y is Doppler frequency, was obtained. The estimate may be assumed to differ from a physical map of the lunar surface for reasons pointed out in Chap. II, Sect. A. Furthermore, aside from equipment limitations, noise and smoothing will give rise to errors, as discussed in Chap. II, Sects. C and D; and Chap. III, Sect. D.

In the following we apply the theory developed in the preceding sections to obtain the optimum waveform and receiver weighting function appropriate for mapping the lunar surface with a radar of the same carrier frequency (440 Mc) and limit on observation time (T_{max} = 9 sec), as were reported for the Millstone experiment. The observation-time limit is set by r-f oscillator stability. From the Millstone data[11] it is clear that the scatterers on the moon's surface are confined to a range interval equal to the moon radius, which corresponds to a delay time or x interval of 23 msec. Furthermore, the scatterers are confined to a range-rate interval, at maximum libration rate, that corresponds, at 440 Mc carrier frequency, to a Doppler frequency or y interval of about 76 rad per sec. Consequently, the samples $\psi(x, 2\pi k/T)$ may be expected to be zero, or very nearly zero, for x outside D = 23 msec; and for $2\pi k/T$ outside B = 76 rad per sec. Thus, a BD scatterer density is at least approximated; and BD = 1.7 < 2π. Observe that, as long as BD < 2π, the theory of Chap. II, Sects. D, E, and F, is applicable; and the period P may have any value D < P < $2\pi/B$ or 23 msec < P < 83 msec. A period of 30 msec was used in the Millstone experiment.

Suppose that $a(x)$, the function that specifies the smoothing in time delay, is chosen to be a triangular pulse, as shown in Fig. 3 where δ is the width at the half-amplitude points. The function $b(y)$ that specifies the smoothing along the Doppler-frequency axis must be chosen so that its Fourier transform is confined to the interval T. A suitable spike-like form for $b(y)$ is

$$b(y) = \frac{\sin yT/2}{yT/2} \tag{3-65}$$

and for this choice of $b(y)$, the attainable y-axis resolution is limited to that obtaining for $T = T_{max}$.

Having chosen the smoothing functions, we shall apply directly the theory in Sects. E and F to obtain, for any specific P, δ, and T, the optimum transmitted waveform and the corresponding receiver weighting function. We shall also investigate the effect of changes in P, δ, and T on the radar performance.

From Fig. 3 and Eq. 3-55, we see that the $|f_n|^2$ of the optimum transmitted waveform must follow a $(\sin x/x)^2$ envelope such that the time autocorrelation of $f(t)$ is a periodic string of triangular pulses of the form shown in Fig. 3. Since the phases of the transmitted harmonics f_n are arbitrary, the above condition can be satisfied by a wide class of transmitted waveforms, among which is the periodic train of rectangular pulses of width δ and period P. The relative simplicity of the transmitter and the receiver are important advantages of the periodic pulse waveform. However, other waveforms, whose f_n have different phase angles, may result in lower peak power requirements if the conditions of the theory can be met in the construction

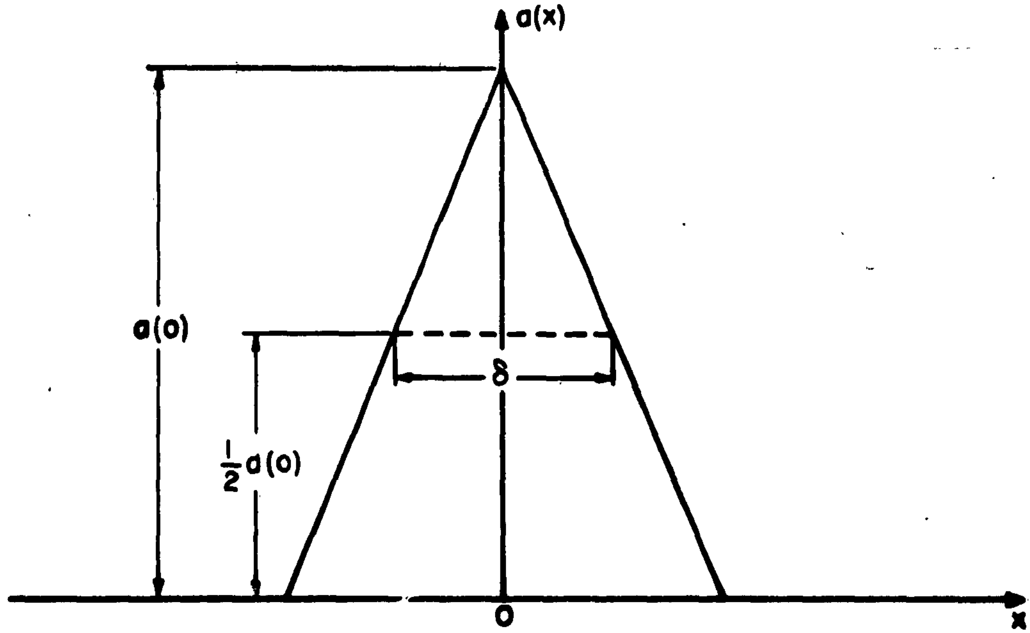

Fig. 3 Choice for a (x)

of the transmitter and receiver. The waveform actually used in the Millstone experiment was an approximately rectangular pulse train with a pulse width $\delta = 500$ μsec and a period $P = 30$ msec.

It follows from Eqs. 3-59, 3-65, and 3-64 that the appropriate receiver is a matched filter with an integration time equal to T. For $T = T_{max} = 9$ sec, such a receiver corresponds very closely to the digital data-processing scheme used in the Millstone experiment.

Thus, if the same smoothing is to be obtained as in the Millstone experiment, the optimum transmitted waveform (aside from the possibility of altering the phases of the transmitted harmonics as discussed above) and the corresponding receiver weighting function obtained on the basis of the dense-scatterer theory are in agreement with the waveform used in the Millstone experiment.

Assuming that $\psi(x,y)$ is a rough scatterer density, we see from Eqs. 3-54, 3-53, and 3-56 that, for a given average power, the average-signal-to-average-noise ratio is independent of the duration of the observation interval T and the smoothing in Doppler frequency $b(y)$. Thus, Doppler-frequency resolution can be increased without cost in signal-to-noise ratio by increasing T from any small value to the allowable maximum. The interval $T_{max} = 9$ sec used in the Millstone experiment may thus be considered optimum in view of the equipment limitations. In contrast, as the width of the smoothing function $a(x)$ decreases, the average-signal-to-average-noise ratio decreases. The choice of $\delta = 500$ μsec in the Millstone experiment is thus a compromise between the greater resolution a shorter pulse would provide and the cost in signal-to-noise ratio (for fixed average transmitted power).

As long as $D < P < 2\pi/B$, we note that: (1) the samples $\psi(x, 2\pi k/T)$ are zero outside an interval smaller than $2\pi/P$ along the y-axis, and $\psi(x,y)$ is zero outside an interval smaller than P along the x-axis; and that (2) the set of $\theta_n(x,y)$ is complete (assuming $f_n \neq 0$ for all n) for x inside an interval of length P and for samples at $y = 2\pi k/T$ inside an interval of length $2\pi/P$. Therefore, if the lunar surface is characterizable by a rough scatterer density, Eq. 3-54 applies. As a result, the average-signal-to-average-noise ratio is independent of the period P for $D < P < 2\pi/B$. Furthermore, the resolution is also independent of P for $D < P < 2\pi/B$.

CHAPTER IV

CONCLUSIONS

A. GENERAL THEORY

There are two basic limitations on radar performance in dense-scatterer applications:

1. The incompleteness of the set of characteristic functions $\theta_n(x,y)$. This incompleteness results in an ambiguous component of the scatterer density, $\psi_\perp(x,y)$, which makes no contribution to the radar return and hence, even in the absence of noise, cannot be deduced from the received waveform. Only the unambiguous component, $\psi_{||}(x,y)$, can be uniquely determined from the noise-free radar return.

2. In the presence of additive, stationary, white, Gaussian noise, only an estimate of a smoothed form of $\psi(x,y)$ can be obtained and, then, only if the desired smoothing function is expandable in terms of the $\theta_n(x,y)$. As the degree of smoothing is increased, the signal-to-noise ratio on the radar display increases; but the sharp details of $\psi(x,y)$ (or $|\psi(x,y)|$) are smoothed out. Any radar design must be a compromise between a loss of detail in excess of that resulting from the incompleteness of the $\theta_n(x,y)$ and the signal-to-noise ratio on the display.

B. DELAY-DOPPLER THEORY

Time delay and Doppler frequency constitute a pair of coordinates for which a scatterer-density function can be defined so that its determination or estimation may be taken as the objective of the radar system.

If all the time delays are known a priori to be confined to a finite interval D, and if the observation time is confined to a finite interval, then:

1. Only periodic waveforms need be considered as possible transmitted waveforms.
2. A transmitted waveform of period P less than D will give rise to an ambiguous component of the scatterer density.
3. Only the $\psi_T(x,y)$ component of the scatterer density, that is, the component that can be expressed as a sin x/x interpolation of samples spaced at intervals $2\pi/T$ along y (see Eq. 3-11), contributes to the radar return.

If a periodic transmitted waveform of period P equal to D is used, and if it is known a priori that the samples of $\psi_T(x,y)$ at intervals $2\pi/T$ along y are zero for y outside an interval B, then $\psi_T(x,y)$ can be deduced unambiguously from the noise-free radar return if, and only if,

1. $BD < 2\pi$
2. All harmonics of the transmitted waveform are different from zero.

If the foregoing conditions are satisfied and if a(x), specifying the smoothing along the x-axis, is a realizable auto-correlation function, then, for a given transmitted average power the output signal-to-noise ratio is a maximum when the time autocorrelation function of the transmitted waveform is a periodic string of pulses of the form of a(x). The receiver weighting function appropriate for such a waveform resembles closely the matched filter.

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Radar is considered as a means of producing a map in two coordinates (x and y) of a dense distribution of scatterers. It is assumed that a complex unit-scatterer return $h(t; x, y)$ and a complex scatterer-density function $\psi(x, y)$ can be defined such that: (a) $h(t; x, y)$ depends only on the radar; (b) $\psi(x, y)$ depends only on the scatterer distribution; (c) the complex video representation of the echo signal is

$s(t) = \int_{-\infty}^{\infty} \psi(x, y)\, h(t; x, y)\,dx\,dy$; and (d) an x-y

display of $\psi(x, y)$ approximates, in some practically useful way, a map of the distribution

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A RADAR THEORY APPLICABLE TO DENSE SCATTERER DISTRIBUTIONS by Arthur Krinitz. January 1962. 65p. incl. illus. Contract AF-33(616)-5489, Task 50688, MIT Project DSR 7848 and Contract AF-33(657)-7644, Task 414409, MIT Project DSR 9040. (Report ESL-R-131) Unclassified Report

Radar is considered as a means of producing a map in two coordinates (x and y) of a dense distribution of scatterers. It is assumed that a complex unit-scatterer return $h(t; x, y)$ and a complex scatterer-density function $\psi(x, y)$ can be defined such that: (a) $h(t; x, y)$ depends only on the radar; (b) $\psi(x, y)$ depends only on the scatterer distribution; (c) the complex video representation of the echo signal is

$s(t) = \int_{-\infty}^{\infty} \psi(x, y)\, h(t; x, y)\,dx\,dy$; and (d) an x-y

display of $\psi(x, y)$ approximates, in some practically useful way, a map of the distribution

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of scattering objects in the x-y plane. The above conditions are satisfied if x and y are taken to be time delay and Doppler frequency. Determination of $\psi(x, y)$ is assumed to be the objective of the radar.

A series expansion of $h(t; x, y)$ is obtained in terms of a set of functions, $\phi_n(t)$, which are orthonormal over the receiver operating time interval T, and a set of functions $\theta_n(x, y)$ which are orthonormal over a finite region R of the x-y plane. If the only *a priori* information is that $\psi(x, y)$ is zero outside the region R, then, even in the absence of noise, only the component of $\psi(x, y)$ which is representable as a linear combination of the $\theta_n(x, y)$ can be deduced from the radar return.

A smoothed form $\psi_s(x, y)$ of $\psi(x, y)$ is defined as the convolution of $\psi(x, y)$ with a spike-like smoothing function. In the presence of additive, stationary, white Gaussian noise, a maximum likelihood estimate of $\psi_s(x, y)$ can be obtained provided the $\theta_n(x, y)$ are sufficiently complete to permit expanding the desired smoothing function in terms of the $\theta_n(x, y)$. The appropriate receiver is not, in general, a matched filter. The cross section of the smoothing-function spike must be chosen to compromise between loss of detail in $\psi_s(x, y)$ and reduction of the signal-to-noise ratio.

The general theory is applied to the case where x is time delay and y is Doppler frequency.

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LIST OF REPORTS PUBLISHED ON THIS CONTRACT

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7848-R-2	AD 210248	Unclassified	Analysis of Magnetic Amplifiers	Johannessen, P.	10/58
7848-R-3	AD 306104	Confidential	Radar System and Component Research 1958 Annual Report	Staff	11/58
7848-R-3a		Unclassified	Unclassified Radar System and Component Research	Staff	11/58
7848-R-4	AD 210245	Unclassified	Video-Frequency Noise Measurements on Microwave Crystal Mixers	Newberg, I. L.	8/59
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ESL-AR-104(2)	AD 325991	Confidential	(Unclassified Title) Investigation of New Radar Components and Techniques--1960 Annual Report (Vol. II)	Staff	9/59-8/60
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